

MA2001

MULTI-VARIABLE CALCULUS AND LINEAR ALGEBRA

Chapter 4

Vector Differential Calculus

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Chapter 4: Vector Differential Calculus

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1 Differentiation of a Vector Function

Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ be a **vector function** of a scalar variable t on an interval I . Then $\vec{r}(t)$ traces a curve in $\mathbb{R}^3$ with an orientation.

The **derivative** of $\vec{r}(t)$ with respect to t , denoted by $\frac{d\vec{r}}{dt}$ or $\vec{r}'(t)$, is defined as:

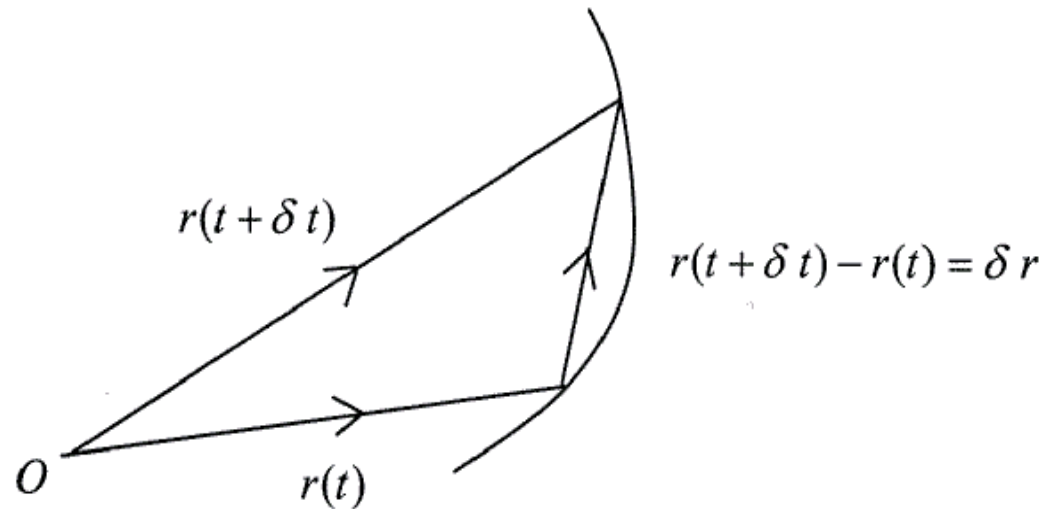
$$\begin{aligned}\frac{d\vec{r}}{dt} &= \lim_{\delta t \rightarrow 0} \left[\frac{\vec{r}(t + \delta t) - \vec{r}(t)}{\delta t} \right] \\ &= \underbrace{\left[\lim_{\delta t \rightarrow 0} \frac{x(t + \delta t) - x(t)}{\delta t} \right]}_{=\frac{dx}{dt}} \vec{i} + \underbrace{\left[\lim_{\delta t \rightarrow 0} \frac{y(t + \delta t) - y(t)}{\delta t} \right]}_{=\frac{dy}{dt}} \vec{j} + \underbrace{\left[\lim_{\delta t \rightarrow 0} \frac{z(t + \delta t) - z(t)}{\delta t} \right]}_{=\frac{dz}{dt}} \vec{k} \\ &= \frac{dx}{dt} \vec{i} + \frac{dy}{dt} \vec{j} + \frac{dz}{dt} \vec{k}\end{aligned}$$

$$\therefore \boxed{\frac{d\vec{r}}{dt} = \frac{dx}{dt} \vec{i} + \frac{dy}{dt} \vec{j} + \frac{dz}{dt} \vec{k}} \quad \text{or written as} \quad \boxed{\vec{r}'(t) = x'(t) \vec{i} + y'(t) \vec{j} + z'(t) \vec{k}}.$$

Therefore, the vector function $\vec{r}(t)$ is differentiable at $t = a$ if $x(t)$, $y(t)$ and $z(t)$ are differentiable at $t = a$. We say that $\vec{r}(t)$ is differentiable if it is differentiable at every point of its domain.

Position vector, velocity vector and acceleration vector

If $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ is the **position vector** of a moving point on a smooth curve at time t , then



- $\vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{\delta t \rightarrow 0} \frac{\delta \vec{r}}{\delta t}$ represents the **velocity** $\vec{v}(t)$ of the moving point and is seen to be a vector tangential to the path of the moving point at $\vec{r}(t)$, and
- $\vec{r}''(t) = \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt} \left(\frac{d\vec{r}}{dt} \right) = \frac{d\vec{v}}{dt}$ represents the **acceleration** $\vec{a}(t)$ of the moving point.

Example 1

A particle moves in such a way that its position vector at time t is

$$\vec{r}(t) = e^{-t}\vec{i} + 2 \cos 3t \vec{j} + 3 \sin 3t \vec{k}.$$

Determine its velocity $\vec{v}$ and acceleration $\vec{a}$ at time $t = 0$. Find the magnitudes and unit directions of $\vec{v}$ and $\vec{a}$ at $t = 0$.

Solution:

The velocity of the particle is

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -e^{-t}\vec{i} - 6 \sin 3t \vec{j} + 9 \cos 3t \vec{k}.$$

At time $t = 0$, $\vec{v}(0) = -\vec{i} + 9\vec{k}$.

The magnitude of $\vec{v}(0)$ is

$$|\vec{v}(0)| = \sqrt{(-1)^2 + 9^2} = \sqrt{82}$$

and the unit direction of $\vec{v}(0)$ is

$$\frac{\vec{v}(0)}{|\vec{v}(0)|} = \frac{1}{\sqrt{82}} (-\vec{i} + 9\vec{k}).$$

The acceleration of the particle is

$$\begin{aligned}\vec{a}(t) &= \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}(-e^{-t}\vec{i} - 6\sin 3t\vec{j} + 9\cos 3t\vec{k}) \\ &= e^{-t}\vec{i} - 18\cos 3t\vec{j} - 27\sin 3t\vec{k}.\end{aligned}$$

At time $t = 0$, $\vec{a}(0) = \vec{i} - 18\vec{j}$.

The magnitude of $\vec{a}(0)$ is

$$|\vec{a}(0)| = \sqrt{1^2 + (-18)^2} = \sqrt{325}$$

and the unit direction of $\vec{a}(0)$ is

$$\frac{\vec{a}(0)}{|\vec{a}(0)|} = \frac{1}{\sqrt{325}}(\vec{i} - 18\vec{j}).$$

Length of curve and Arc length function

Let C be a space curve with a parametric representation as

$$C : \vec{r}(t) = x(t) \vec{i} + y(t) \vec{j} + z(t) \vec{k} \quad (a \leq t \leq b),$$

where the number t is called a parameter.

C is a smooth space curve if $x(t)$, $y(t)$, $z(t)$ (where $a \leq t \leq b$) are continuous and $x'(t)$, $y'(t)$, $z'(t)$ (where $a \leq t \leq b$) exist and are continuous. Here, $x'(a)$, $y'(a)$, $z'(a)$ denote the right-hand derivatives, and $x'(b)$, $y'(b)$, $z'(b)$ denote the left-hand derivatives.

In addition, $\frac{d\vec{r}(t)}{dt} \neq \vec{0}$ for $a \leq t \leq b$.

We call $\frac{d\vec{r}(t)}{dt}$ (or $\vec{r}'(t)$) a tangent vector of C at point P because it has the direction of the tangent.

The length l of the curve C is

$$l = \int_a^b \left| \frac{d\vec{r}(t)}{dt} \right| dt.$$

This is a positive constant. If we replace the dummy variable t in the above definite integral by u and then replace the upper limit b by the variable t , we obtain the **arc length function**:

$$s(t) = \int_a^t \left| \frac{d\vec{r}(u)}{du} \right| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du, \quad a \leq t \leq b.$$

Note that $s(t)$ is a function of t .

It represents the distance travelled by the moving point at time t .

Geometrically, it also represents the length of the part of C represented on $a \leq u \leq t$.

Clearly, $s(a) = 0$, $s(b) = l$.

Rules of differentiation for vector functions

Let $\vec{a}(t)$ and $\vec{b}(t)$ be differentiable vector-valued functions of t , c be a constant, and $f(t)$ be a differentiable scalar function. Then

$$(i) \quad \frac{d[c \vec{a}(t)]}{dt} = c \frac{d[\vec{a}(t)]}{dt}$$

$$(ii) \quad \frac{d[\vec{a}(t) \pm \vec{b}(t)]}{dt} = \frac{d[\vec{a}(t)]}{dt} \pm \frac{d[\vec{b}(t)]}{dt}$$

$$(iii) \quad \frac{d[f(t) \vec{a}(t)]}{dt} = \frac{d[f(t)]}{dt} \vec{a}(t) + f(t) \frac{d[\vec{a}(t)]}{dt}$$

$$(iv) \quad \frac{d[\vec{a}(t) \cdot \vec{b}(t)]}{dt} = \frac{d[\vec{a}(t)]}{dt} \cdot \vec{b}(t) + \vec{a}(t) \cdot \frac{d[\vec{b}(t)]}{dt}$$

$$(v) \quad \frac{d[\vec{a}(t) \times \vec{b}(t)]}{dt} = \frac{d[\vec{a}(t)]}{dt} \times \vec{b}(t) + \vec{a}(t) \times \frac{d[\vec{b}(t)]}{dt}$$

Remarks:

- (a) If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ where a_1, a_2, a_3 are functions of several variables $t_1, t_2, \dots, t_n$, then the partial derivatives

$$\frac{\partial \vec{a}}{\partial t_i} = \frac{\partial a_1}{\partial t_i}\vec{i} + \frac{\partial a_2}{\partial t_i}\vec{j} + \frac{\partial a_3}{\partial t_i}\vec{k}, \quad 1 \leq i \leq n$$

may be defined in the obvious way.

- (b) Also, vector integration may be defined as follows.

Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$. Then

$$\int \vec{r}(t) dt = \vec{R}(t) + \vec{C},$$

where $\frac{d[\vec{R}(t)]}{dt} = \vec{r}(t)$, and $\vec{C}$ is any constant vector.

Here,

$$\int \vec{r}(t) dt = \left[\int x(t) dt \right] \vec{i} + \left[\int y(t) dt \right] \vec{j} + \left[\int z(t) dt \right] \vec{k} + \vec{C}.$$

Example 2

If a particle moves with velocity $\vec{v}(t) = 2 \sin t \vec{i} + (\cos t - 1) \vec{j} + 6t \vec{k}$ and $\vec{r} = \vec{0}$ at $t = 0$, determine its position vector $\vec{r}(t)$ at time t .

Solution

$$\begin{aligned}\vec{r}(t) &= \int \frac{d\vec{r}}{dt} dt \\&= \int \vec{v}(t) dt \\&= \int [2 \sin t \vec{i} + (\cos t - 1) \vec{j} + 6t \vec{k}] dt \\&= \left[\int 2 \sin t dt \right] \vec{i} + \left[\int (\cos t - 1) dt \right] \vec{j} + \left[\int 6t dt \right] \vec{k} \\&= (-2 \cos t) \vec{i} + (\sin t - t) \vec{j} + 3t^2 \vec{k} + \vec{C}\end{aligned}$$

$$\text{At } t = 0, \vec{0} = -2\vec{i} + \vec{C} \Rightarrow \vec{C} = 2\vec{i}.$$

$$\therefore \vec{r}(t) = (2 - 2 \cos t) \vec{i} + (\sin t - t) \vec{j} + 3t^2 \vec{k}.$$

□

Example 3

Let $\vec{r}$ in polar form be the position vector of a moving particle in $\mathbb{R}^2$. Find the velocity and the acceleration of $\vec{r}$ in polar form.

Solution

Representing $\vec{r}$ in polar form, we write

$$\vec{r} = r \cos \theta \vec{i} + r \sin \theta \vec{j} = r \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = r \vec{R},$$

where $\vec{R} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ is a unit vector in the direction of $\vec{r}$, and $r = |\vec{r}|$ is the magnitude of $\vec{r}$.

The **velocity** $\vec{v}$ of the moving particle in polar form is

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(r\vec{R})}{dt} = \frac{dr}{dt} \vec{R} + r \frac{d\vec{R}}{dt} = \frac{dr}{dt} \vec{R} + r \frac{d\vec{R}}{d\theta} \frac{d\theta}{dt}.$$

$$\text{Let } \vec{P} = \frac{d\vec{R}}{d\theta} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}.$$

Note that $\vec{R} \cdot \vec{P} = -\cos \theta \sin \theta + \sin \theta \cos \theta = 0$, so $\vec{R}$ is perpendicular to $\vec{P} = \frac{d\vec{R}}{d\theta}$.

Then $\vec{v} = \frac{dr}{dt} \vec{R} + r \frac{d\theta}{dt} \vec{P}$, i.e. $\vec{v}$ is a linear combination of two perpendicular unit vectors $\vec{R}$, $\vec{P}$.

The **acceleration** $\vec{a}$ of the moving particle in polar form is

$$\begin{aligned}
 \vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt} \left(\frac{dr}{dt} \vec{R} + r \frac{d\theta}{dt} \vec{P} \right) = \frac{d}{dt} \left(\frac{dr}{dt} \vec{R} \right) + \frac{d}{dt} \left(r \frac{d\theta}{dt} \vec{P} \right) \\
 &= \frac{d}{dt} \left(\frac{dr}{dt} \right) \vec{R} + \frac{dr}{dt} \underbrace{\frac{d\vec{R}}{dt}}_{=\frac{d\vec{R}}{d\theta} \frac{d\theta}{dt}} + \frac{dr}{dt} \frac{d\theta}{dt} \vec{P} + r \frac{d}{dt} \left(\frac{d\theta}{dt} \right) \vec{P} + r \frac{d\theta}{dt} \underbrace{\frac{d\vec{P}}{dt}}_{=\frac{d\vec{P}}{d\theta} \frac{d\theta}{dt}} \\
 &= \frac{d^2 r}{dt^2} \vec{R} + \frac{dr}{dt} \underbrace{\frac{d\vec{R}}{d\theta} \frac{d\theta}{dt}}_{=\vec{P}} + \frac{dr}{dt} \frac{d\theta}{dt} \vec{P} + r \frac{d^2 \theta}{dt^2} \vec{P} + r \frac{d\theta}{dt} \frac{d\vec{P}}{d\theta} \frac{d\theta}{dt} \\
 &= \frac{d^2 r}{dt^2} \vec{R} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \vec{P} + r \frac{d^2 \theta}{dt^2} \vec{P} + r \left(\frac{d\theta}{dt} \right)^2 \underbrace{\frac{d\vec{P}}{d\theta}}_{=-\vec{R}} \\
 &= \left[\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] \vec{R} + \left(2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2} \right) \vec{P}
 \end{aligned}$$

where $\frac{d\vec{P}}{d\theta} = \frac{d}{d\theta} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} = \begin{pmatrix} -\cos \theta \\ -\sin \theta \end{pmatrix} = -\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = -\vec{R}.$

2 Vector Fields

A **vector field** on $\mathbb{R}^3$ is a **vector function** $\vec{v}$ that assigns to each point (x, y, z) a vector of $\mathbb{R}^3$ given by

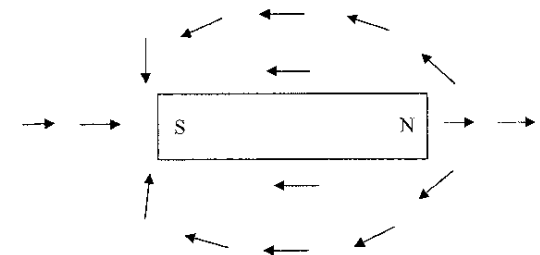
$$\vec{v}(x, y, z) = v_1(x, y, z) \vec{i} + v_2(x, y, z) \vec{j} + v_3(x, y, z) \vec{k},$$

where $v_1(x, y, z)$, $v_2(x, y, z)$ and $v_3(x, y, z)$ are scalar functions.

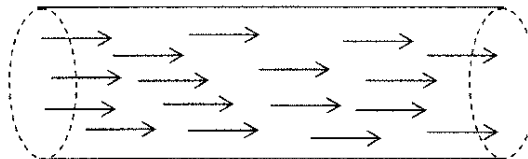
Examples of vector fields

(i) A magnetic field $\vec{B}$ in a region of space,

$$\vec{B} = B_1 \vec{i} + B_2 \vec{j} + B_3 \vec{k}.$$



(ii) The velocity field of water flowing in a pipe, $\vec{v}(x, y, z)$.



(iii) The earth's gravitational field.

3 Gradient of a Scalar Field

Recall from Chapter 2 that for any differentiable scalar field $f(x, y, z)$, its **gradient**, denoted by **$\text{grad } f$** , is a vector defined as

$$\text{grad } f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}.$$

This is also an example of a vector field which gives the magnitude and direction of the **greatest rate of change** of f at a point.

The **vector differential operator**, denoted by ∇ , is defined as

$$\nabla \equiv \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$$

(analogous to $D \equiv \frac{d}{dx}$), and is read as “del” (or “nabla”).

In terms of the operator ∇ , if we write $\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) f$, then

$$\boxed{\text{grad } f = \nabla f}.$$

Geometric interpretation of $\text{grad } f$

Let S be a surface of $\mathbb{R}^3$ defined by the equation $f(x, y, z) = c$, where c is a constant. If $P(x_0, y_0, z_0)$ is any point on S , then the gradient vector $\nabla f(x_0, y_0, z_0)$ is **normal to the surface** S at P .

We may also apply the operator ∇ to a vector field

$$\vec{v}(x, y, z) = v_1(x, y, z) \vec{i} + v_2(x, y, z) \vec{j} + v_3(x, y, z) \vec{k}$$

through dot product ($\nabla \cdot \vec{v}$ = divergence of $\vec{v}$) or cross product ($\nabla \times \vec{v}$ = curl of $\vec{v}$). These topics will be discussed in Section 4 and 5, respectively.

4 Divergence of a Vector Field

Let $\vec{v}(x, y, z) = v_1(x, y, z) \vec{i} + v_2(x, y, z) \vec{j} + v_3(x, y, z) \vec{k}$ be a vector field such that v_1 , v_2 and v_3 are all differentiable scalar functions.

The divergence of $\vec{v}$, denoted by $\text{div } \vec{v}$, is defined as

$$\boxed{\text{div } \vec{v} \equiv \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}}.$$

In terms of the operator ∇ , we may write

$$\begin{aligned} \nabla \cdot \vec{v} &\equiv \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot [v_1(x, y, z) \vec{i} + v_2(x, y, z) \vec{j} + v_3(x, y, z) \vec{k}] \\ &= \frac{\partial v_1(x, y, z)}{\partial x} + \frac{\partial v_2(x, y, z)}{\partial y} + \frac{\partial v_3(x, y, z)}{\partial z} \\ &= \text{div } \vec{v} \end{aligned}$$

$$\therefore \boxed{\text{div } \vec{v} = \nabla \cdot \vec{v}}$$

Note that $\nabla \cdot \vec{v}$ is a scalar function.

Also, $\nabla \cdot \vec{v} \neq \vec{v} \cdot \nabla$. (Here, $\nabla \cdot \vec{v}$ is a scalar field, while $\vec{v} \cdot \nabla$ is a scalar differential operator.)

Example 4

Let $\vec{v} = x^2z \vec{i} - 2y^3z \vec{j} + xyz^2 \vec{k}$, find $\operatorname{div} \vec{v}$.

Solution:

$$\begin{aligned}\operatorname{div} \vec{v} &= \nabla \cdot \vec{v} \\&= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (x^2z \vec{i} - 2y^3z \vec{j} + xyz^2 \vec{k}) \\&= \frac{\partial(x^2z)}{\partial x} + \frac{\partial(-2y^3z)}{\partial y} + \frac{\partial(xyz^2)}{\partial z} \\&= 2xz - 6y^2z + 2xyz\end{aligned}$$

□

Physical interpretation of $\operatorname{div} \vec{v}$

Roughly speaking, the divergence of a vector field $\vec{v}$ at point P is a measure of the rate at which the field “diverges” or “spreads” away from P .

The divergence of a vector field $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ can be considered as the net outward flux (per unit volume) from a small closed surface area containing a point (x, y, z) in the flow of an incompressible fluid (for example water) as volume shrinks to zero, where $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is the velocity of the fluid.

As a measure of the net outward flux per unit volume at a point (x, y, z) , **positive divergence** indicates a source of fluid at the point (where the flux is directed **outward**) whilst **negative divergence** indicates a sink (where the flux is directed **inward**).

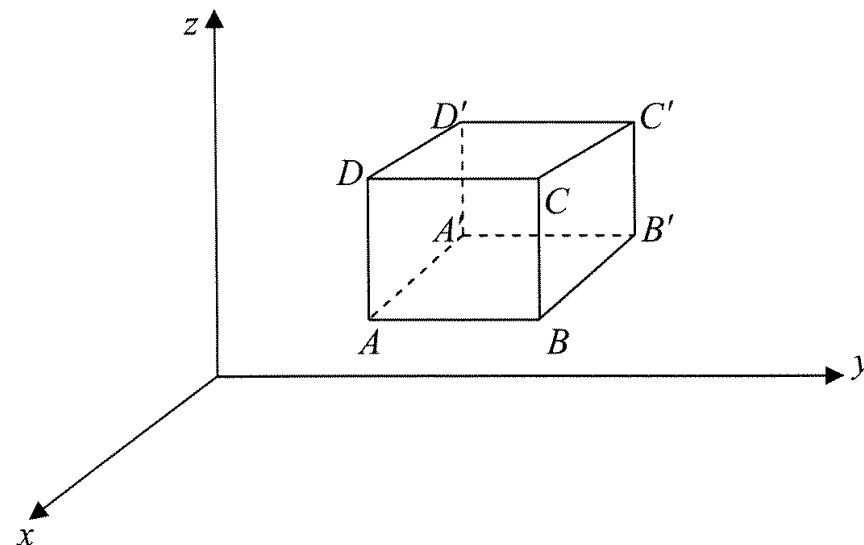
For your reference: (p.22-24)

The idea of divergence is probably most easily illustrated by considering the motion of an incompressible fluid (for example water) where we take the vector field to be the velocity $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ of the fluid.

Consider a small imaginary rectangular box of dimensions δx , δy and δz centered at (x, y, z) in the fluid.

We consider the flow of fluid through this box. The volume of fluid crossing a surface in unit time is an example of flux. Taking the midpoint of a face as a representative point, the flux through the face $ABCD$ out of the box is $v_1 \left(x + \frac{\delta x}{2}, y, z\right) \delta y \delta z$ (v_2 and v_3 are parallel to $ABCD$ and make no contribution).

Similarly, the flux through face $A'B'C'D'$ into the box is $v_1 \left(x - \frac{\delta x}{2}, y, z\right) \delta y \delta z$.



The net flux out of the box in the x -direction is

$$\begin{aligned}
 & \left[v_1 \left(x + \frac{\delta x}{2}, y, z \right) - v_1 \left(x - \frac{\delta x}{2}, y, z \right) \right] \delta y \delta z \\
 &= \left[\left(v_1(x, y, z) + \frac{\left(\frac{\delta x}{2}\right)}{1!} \frac{\partial v_1(x, y, z)}{\partial x} + \frac{\left(\frac{\delta x}{2}\right)^2}{2!} \frac{\partial^2 v_1(x, y, z)}{\partial x^2} + \frac{\left(\frac{\delta x}{2}\right)^3}{3!} \frac{\partial^3 v_1(x, y, z)}{\partial x^3} + \dots \right) \right. \\
 & \quad \left. - \left(v_1(x, y, z) + \frac{\left(-\frac{\delta x}{2}\right)}{1!} \frac{\partial v_1(x, y, z)}{\partial x} + \frac{\left(-\frac{\delta x}{2}\right)^2}{2!} \frac{\partial^2 v_1(x, y, z)}{\partial x^2} + \frac{\left(-\frac{\delta x}{2}\right)^3}{3!} \frac{\partial^3 v_1(x, y, z)}{\partial x^3} + \dots \right) \right] \delta y \delta z \\
 &= \left[\frac{2}{1!} \left(\frac{\delta x}{2} \right) \frac{\partial v_1(x, y, z)}{\partial x} + \frac{2}{3!} \left(\frac{\delta x}{2} \right)^3 \frac{\partial^3 v_1(x, y, z)}{\partial x^3} + \dots \right] \delta y \delta z \\
 &= \left[\frac{\partial v_1(x, y, z)}{\partial x} + \frac{(\delta x)^2}{3! 2^2} \frac{\partial^3 v_1(x, y, z)}{\partial x^3} + \dots \right] \delta x \delta y \delta z
 \end{aligned}$$

Similarly, by considering the two other pairs of faces, the total flux out of the box is

$$\begin{aligned}
 & \left[\frac{\partial v_1(x, y, z)}{\partial x} + \frac{\partial v_2(x, y, z)}{\partial y} + \frac{\partial v_3(x, y, z)}{\partial z} \right] \delta x \delta y \delta z \\
 & + \left[\frac{(\delta x)^2}{3! 2^2} \frac{\partial^3 v_1(x, y, z)}{\partial x^3} + \frac{(\delta y)^2}{3! 2^2} \frac{\partial^3 v_2(x, y, z)}{\partial y^3} + \frac{(\delta z)^2}{3! 2^2} \frac{\partial^3 v_3(x, y, z)}{\partial z^3} + \dots \right] \delta x \delta y \delta z
 \end{aligned}$$

If we consider the flux per unit volume, we have

$$\left[\frac{\partial v_1(x, y, z)}{\partial x} + \frac{\partial v_2(x, y, z)}{\partial y} + \frac{\partial v_3(x, y, z)}{\partial z} \right] + \left[\frac{(\delta x)^2}{3! 2^2} \frac{\partial^3 v_1(x, y, z)}{\partial x^3} + \frac{(\delta y)^2}{3! 2^2} \frac{\partial^3 v_2(x, y, z)}{\partial y^3} + \frac{(\delta z)^2}{3! 2^2} \frac{\partial^3 v_3(x, y, z)}{\partial z^3} + \dots \right]$$

Taking the limit as the volume of the box shrinks to zero, we obtain

$$\begin{aligned} \operatorname{div} \vec{v} &= \lim_{\substack{\delta x \rightarrow 0 \\ \delta y \rightarrow 0 \\ \delta z \rightarrow 0}} \left[\frac{\partial v_1(x, y, z)}{\partial x} + \frac{\partial v_2(x, y, z)}{\partial y} + \frac{\partial v_3(x, y, z)}{\partial z} \right] \\ &\quad + \lim_{\substack{\delta x \rightarrow 0 \\ \delta y \rightarrow 0 \\ \delta z \rightarrow 0}} \left[\frac{(\delta x)^2}{3! 2^2} \frac{\partial^3 v_1(x, y, z)}{\partial x^3} + \frac{(\delta y)^2}{3! 2^2} \frac{\partial^3 v_2(x, y, z)}{\partial y^3} + \frac{(\delta z)^2}{3! 2^2} \frac{\partial^3 v_3(x, y, z)}{\partial z^3} + \dots \right] \\ &= \frac{\partial v_1(x, y, z)}{\partial x} + \frac{\partial v_2(x, y, z)}{\partial y} + \frac{\partial v_3(x, y, z)}{\partial z} \end{aligned}$$

Definition

A vector field $\vec{v}$ is said to be **solenoidal** if

$$\boxed{\operatorname{div} \vec{v} = 0}.$$

That is, no net outflow or inflow of fluid.

Example 5

The electric field $\vec{E}$ at a point due to a point charge Q at the origin is given by

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3},$$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ ($\vec{r} \neq \vec{0}$) is the position vector of the point, $r = |\vec{r}|$ is the length of $\vec{r}$, and ϵ_0 is a constant. Show that $\vec{E}$ is solenoidal (for $\vec{r} \neq \vec{0}$).

Solution

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3} = \frac{Q}{4\pi\epsilon_0} \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

Then

$$\operatorname{div} \vec{E} = \nabla \cdot \vec{E}$$

$$= \frac{Q}{4\pi\epsilon_0} \left[\frac{\partial}{\partial x} \left(\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) + \frac{\partial}{\partial y} \left(\frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) + \frac{\partial}{\partial z} \left(\frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) \right]$$

$$= \frac{Q}{4\pi\epsilon_0} \left[\frac{y^2 + z^2 - 2x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} + \frac{x^2 + z^2 - 2y^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} + \frac{x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \right]$$

$$= 0 \text{ (except at the origin)}$$

Hence, $\vec{E}$ is solenoidal (for $\vec{r} \neq \vec{0}$).

(There is no electric charge at any point except the origin.)

□

5 Curl of a Vector Field

Let $\vec{v}(x, y, z) = v_1(x, y, z) \vec{i} + v_2(x, y, z) \vec{j} + v_3(x, y, z) \vec{k}$ be a vector field in $\mathbb{R}^3$ such that v_1 , v_2 and v_3 are all differentiable. The curl of $\vec{v}$, denoted by ***curl*** $\vec{v}$, is defined as

$$\begin{aligned} \text{curl } \vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \vec{i} \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) - \vec{j} \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right) + \vec{k} \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \end{aligned}$$

In terms of the operator ∇ , we have

$$\boxed{\text{curl } \vec{v} = \nabla \times \vec{v}}.$$

Note that $\nabla \times \vec{v}$ is a vector function.

Example 6

If $\vec{v} = x^2z \vec{i} - 2y^3z^2 \vec{j} + xyz^2 \vec{k}$, find $\text{curl } \vec{v}$.

Solution

$$\text{curl } \vec{v} = \nabla \times \vec{v}$$

$$\begin{aligned}
 &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2y^3z^2 & xyz^2 \end{vmatrix} \\
 &= \vec{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2y^3z^2 & xyz^2 \end{vmatrix} - \vec{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x^2z & xyz^2 \end{vmatrix} + \vec{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2z & -2y^3z^2 \end{vmatrix} \\
 &= \left[\frac{\partial(xyz^2)}{\partial y} - \frac{\partial(-2y^3z^2)}{\partial z} \right] \vec{i} - \left[\frac{\partial(xyz^2)}{\partial x} - \frac{\partial(x^2z)}{\partial z} \right] \vec{j} + \left[\frac{\partial(-2y^3z^2)}{\partial x} - \frac{\partial(x^2z)}{\partial y} \right] \vec{k} \\
 &= [xz^2 - (-4y^3z)] \vec{i} - (yz^2 - x^2) \vec{j} + (0 - 0) \vec{k} \\
 &= (xz^2 + 4y^3z) \vec{i} + (x^2 - yz^2) \vec{j}
 \end{aligned}$$



Physical Interpretation of $\text{curl } \vec{v}$

Roughly speaking, $\text{curl } \vec{v}$ at a point P measures the extent to which the vector field $\vec{v}$ “swirls” around P .

For example, the curl of $\vec{v} = v_1(x, y)\vec{i} + v_2(x, y)\vec{j}$ at point $P(x, y)$ can be considered as the limit of the total circulation (net flow around a closed path) per unit area around a small rectangle centered at P . Here,

$$\text{curl } \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1(x, y) & v_2(x, y) & 0 \end{vmatrix} = \left(\frac{\partial v_2(x, y)}{\partial x} - \frac{\partial v_1(x, y)}{\partial y} \right) \vec{k}.$$

(See next page for proof.)

If this quantity is positive/negative/zero, a small paddle wheel placed in the fluid at the point P would rotate anticlockwise/clockwise/remain stationary.

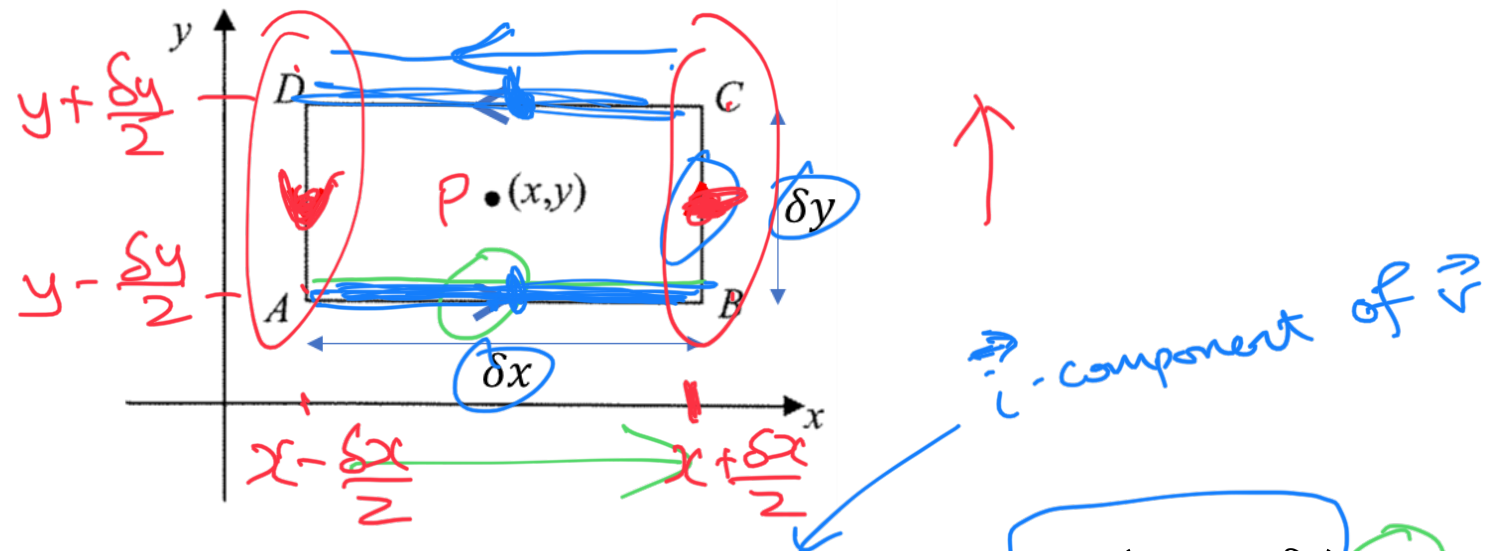


For your reference: (p.30-33)

We shall illustrate the idea of curl using the motion of an incompressible fluid with velocity

$$\vec{v} = v_1(x, y)\vec{i} + v_2(x, y)\vec{j}.$$

Consider just the xy-plane and a small imaginary rectangle of sides δx and δy centered at (x, y) .



Flow along $AB = (\text{length of } AB) \times (\text{velocity component along } AB) = v_1\left(x, y - \frac{\delta y}{2}\right) \delta x$
 and flow along $CD = -v_1\left(x, y + \frac{\delta y}{2}\right) \delta x.$

The sum of flows along **AB + CD** is

$$\begin{aligned}
 & \left[v_1 \left(x, y - \frac{\delta y}{2} \right) - v_1 \left(x, y + \frac{\delta y}{2} \right) \right] \delta x \\
 &= \left[\left(v_1(x, y) + \left(-\frac{\delta y}{2} \right) \frac{\partial v_1(x, y)}{\partial y} + \frac{\left(-\frac{\delta y}{2} \right)^2}{2!} \frac{\partial^2 v_1(x, y)}{\partial y^2} + \dots \right) \right. \\
 & \quad \left. - \left(v_1(x, y) + \left(\frac{\delta y}{2} \right) \frac{\partial v_1(x, y)}{\partial y} + \frac{\left(\frac{\delta y}{2} \right)^2}{2!} \frac{\partial^2 v_1(x, y)}{\partial y^2} + \dots \right) \right] \delta x \\
 &= -\frac{\partial v_1(x, y)}{\partial y} \delta x \delta y - \frac{(\delta y)^3}{3! 2^2} \frac{\partial^3 v_1(x, y)}{\partial y^3} \delta x \delta y - \dots
 \end{aligned}$$

Similarly, the sum of flows along **BC + DA** is

$$\left[v_2 \left(x + \frac{\delta x}{2}, y \right) - v_2 \left(x - \frac{\delta x}{2}, y \right) \right] \delta y = \frac{\partial v_2(x, y)}{\partial x} \delta x \delta y + \frac{(\delta x)^3}{3! 2^2} \frac{\partial^3 v_2(x, y)}{\partial x^3} \delta x \delta y + \dots$$

The total circulation (net flow around a closed path) around the rectangle is

$$\left[\frac{\partial v_2(x, y)}{\partial x} - \frac{\partial v_1(x, y)}{\partial y} \right] \delta x \delta y + \left[\frac{(\delta x)^3}{3! 2^2} \frac{\partial^3 v_2(x, y)}{\partial x^3} - \frac{(\delta y)^3}{3! 2^2} \frac{\partial^3 v_1(x, y)}{\partial y^3} \right] \delta x \delta y + \dots$$

If we consider circulation per unit area, we have

$$\rightarrow \left[\frac{\partial v_2(x, y)}{\partial x} - \frac{\partial v_1(x, y)}{\partial y} \right] + \left[\frac{(\delta x)^3}{3! 2^2} \frac{\partial^3 v_2(x, y)}{\partial x^3} - \frac{(\delta y)^3}{3! 2^2} \frac{\partial^3 v_1(x, y)}{\partial y^3} \right] + \dots$$

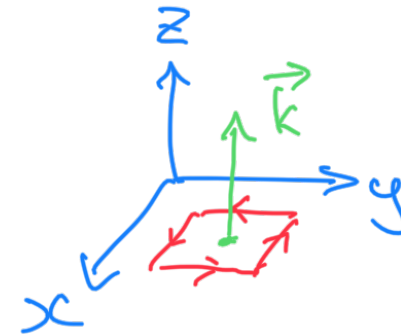
Taking the limit as the area of the rectangle shrinks to zero, we have

$$\lim_{\substack{\delta x \rightarrow 0 \\ \delta y \rightarrow 0}} \left[\frac{\partial v_2(x, y)}{\partial x} - \frac{\partial v_1(x, y)}{\partial y} \right] + \lim_{\substack{\delta x \rightarrow 0 \\ \delta y \rightarrow 0}} \left[\frac{(\delta x)^3}{3! 2^2} \frac{\partial^3 v_2(x, y)}{\partial x^3} - \frac{(\delta y)^3}{3! 2^2} \frac{\partial^3 v_1(x, y)}{\partial y^3} \right] + \dots$$

$$= \frac{\partial v_2(x, y)}{\partial x} - \frac{\partial v_1(x, y)}{\partial y}$$

Hence,

$$\text{curl } \vec{v} = \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \vec{k}$$



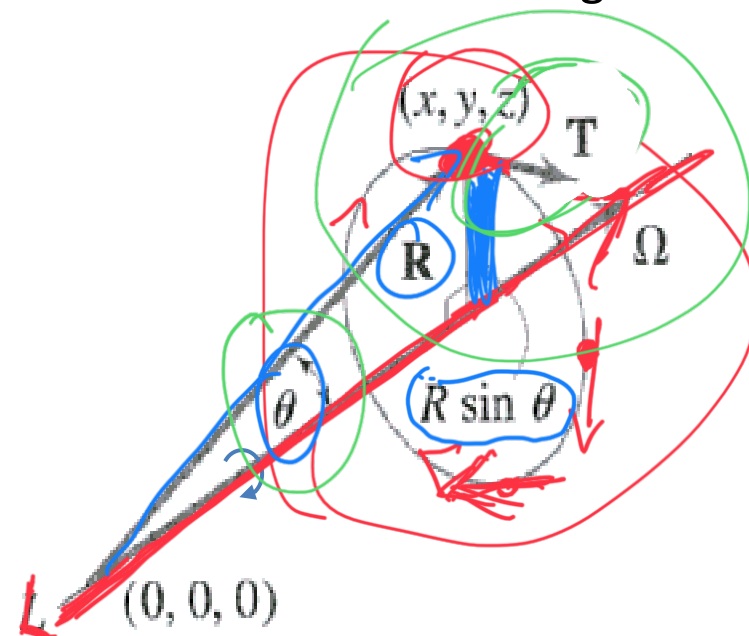
If this quantity is positive/negative/zero, a small paddle wheel placed in the fluid at the point P would rotate anticlockwise/clockwise/remain stationary.

By analogy with a rigid body rotating about an axis with angular velocity ω for which the angular velocity vector is $\omega \vec{n}$, where $\vec{n}$ is a unit vector along the axis of rotation, we may associate with the rotational effect of the fluid by a vector $\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \vec{k}$. This is the third component of $\text{curl } \vec{v}$. Similarly, by considering flow in 3-dimensions we may derive the full expression for $\text{curl } \vec{v}$ which gives a measure of the rotational effect of the fluid flow at a point.

Example 7 (For your reference, p.34-36)

Suppose a body rotates with uniform angular speed ω about a line L . See the figure below.

The angular velocity vector $\vec{\Omega}$ has magnitude (angular speed) ω and is directed along L as a right-handed screw would progress if given the same sense of rotation as the object. Let L go through the origin and let $\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$ be the position vector for any point (x, y, z) on the rotating object.



Let $\vec{T}(x, y, z)$ be the tangential velocity vector. Then

$$|\vec{T}| = \omega |\vec{R}| \sin \theta = |\vec{\Omega}| |\vec{R}| \sin \theta = |\vec{\Omega} \times \vec{R}|,$$

where θ is the angle between $\vec{R}$ and L .

Since $\vec{T}$ and $\vec{\Omega} \times \vec{R}$ are in the same direction, then $\vec{T} = \vec{\Omega} \times \vec{R}$.

magnitude of $\vec{\Omega}$ distance of P from axis

$$\theta = \frac{s}{r}$$

$$s = r\theta$$

Suppose $\vec{\Omega} = a\vec{i} + b\vec{j} + c\vec{k}$. Then

$$\vec{T} = \vec{\Omega} \times \vec{R} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\vec{i} + (cx - az)\vec{j} + (ay - bx)\vec{k}.$$

Thus,

$$\begin{aligned} \text{curl } \vec{T} &= \nabla \times \vec{T} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix} \\ &= \left[\frac{\partial (ay - bx)}{\partial y} - \frac{\partial (cx - az)}{\partial z} \right] \vec{i} - \left[\frac{\partial (ay - bx)}{\partial x} - \frac{\partial (bz - cy)}{\partial z} \right] \vec{j} + \left[\frac{\partial (cx - az)}{\partial x} - \frac{\partial (bz - cy)}{\partial y} \right] \vec{k} \\ &= [a - (-a)]\vec{i} - [(-b) - b]\vec{j} + [c - (-c)]\vec{k} \\ &= 2a\vec{i} + 2b\vec{j} + 2c\vec{k} = 2\vec{\Omega} \end{aligned}$$

$$\therefore \vec{\Omega} = \frac{1}{2} \text{curl } \vec{T} = \frac{1}{2} \nabla \times \vec{T}$$

The angular velocity of a uniformly rotating body is half of the curl of the tangential velocity.

Because of this interpretation, curl was once written rot (for rotation), particularly in British books on mechanics. This is also the motivation for the term **irrotational** for a vector field whose curl is zero.



Definition

A vector field $\vec{v}$ is said to be **irrotational** if

$$\boxed{\text{curl } \vec{v} = \vec{0}}. \text{ everywhere}$$

It does not exhibit any rotational behaviour around a point in space.

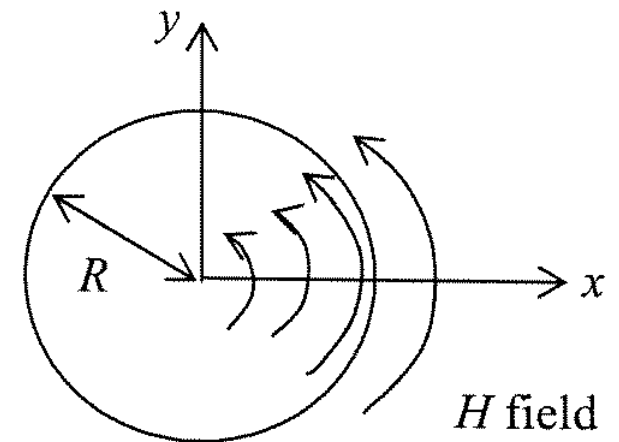
Example 8

Consider a current-carrying conductor with circular cross-section of radius R . Then the magnetic field $\vec{H}$ is

$$\vec{H} = \begin{cases} \frac{-y\vec{i} + x\vec{j}}{2\pi r^2} & \text{outside conductor} \\ \frac{-y\vec{i} + x\vec{j}}{2\pi R^2} & \text{inside conductor} \end{cases},$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$.

Find the curl of $\vec{H}$ (i) outside and (ii) inside the conductor.



Solution

(i) Outside the conductor,

$$\text{curl } \vec{H} = \nabla \times \vec{H}$$

$$\frac{\cancel{x}}{2\pi} (x^2 + y^2)$$

$$\begin{aligned}
 &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{2\pi(x^2 + y^2)} & \frac{x}{2\pi(x^2 + y^2)} & 0 \end{vmatrix} \\
 &= 0\vec{i} + 0\vec{j} + \left[\frac{\partial}{\partial x} \left(\frac{x}{2\pi(x^2 + y^2)} \right) - \frac{\partial}{\partial y} \left(\frac{-y}{2\pi(x^2 + y^2)} \right) \right] \vec{k} \\
 &= \left[\left(\frac{1}{2\pi(x^2 + y^2)} - \frac{2x^2}{2\pi(x^2 + y^2)^2} \right) - \left(\frac{-1}{2\pi(x^2 + y^2)} - \frac{-2y^2}{2\pi(x^2 + y^2)^2} \right) \right] \vec{k} \\
 &= \left(\frac{2}{2\pi(x^2 + y^2)} - \frac{2(x^2 + y^2)}{2\pi(x^2 + y^2)^2} \right) \vec{k} \\
 &= \vec{0}
 \end{aligned}$$

Hence, the magnetic field $\vec{H}$, outside the conductor, is irrotational.

(ii) Inside the conductor,

$$\text{curl } \vec{H} = \nabla \times \vec{H}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{2\pi R^2} & \frac{x}{2\pi R^2} & 0 \end{vmatrix}$$

$$= 0\vec{i} + 0\vec{j} + \left[\frac{\partial}{\partial x} \left(\frac{x}{2\pi R^2} \right) - \frac{\partial}{\partial y} \left(\frac{-y}{2\pi R^2} \right) \right] \vec{k}$$

$$= \left[\frac{1}{2\pi R^2} - \left(\frac{-1}{2\pi R^2} \right) \right] \vec{k}$$

$$= \frac{1}{\pi R^2} \vec{k} \quad (\text{the current density in the conductor})$$

Example 9

The electric field $\vec{E}$ at a point due to a point charge Q at the origin is given by

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3},$$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ ($\vec{r} \neq \vec{0}$) is the position vector of the point, $r = |\vec{r}|$ is the length of $\vec{r}$, and ϵ_0 is a constant. Show that $\vec{E}$ is irrotational.

Solution

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3} = \frac{Q}{4\pi\epsilon_0} \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

Then

$$\text{curl } \vec{E} = \nabla \times \vec{E} = \frac{Q}{4\pi\epsilon_0} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} & \frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} & \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \end{vmatrix}$$

$$\begin{aligned}
&= \frac{Q}{4\pi\epsilon_0} \left\{ \left[\frac{\partial}{\partial y} \left(\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) - \frac{\partial}{\partial z} \left(\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) \right] \vec{i} \right. \\
&\quad - \left[\frac{\partial}{\partial x} \left(\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) - \frac{\partial}{\partial z} \left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) \right] \vec{j} \\
&\quad \left. + \left[\frac{\partial}{\partial x} \left(\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) - \frac{\partial}{\partial y} \left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) \right] \vec{k} \right\}
\end{aligned}$$

$$-\frac{3}{2} z (x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2y$$

$$\begin{aligned}
&= \frac{Q}{4\pi\epsilon_0} \left\{ \left[-\frac{3yz}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \left(-\frac{3yz}{(x^2+y^2+z^2)^{\frac{5}{2}}} \right) \right] \vec{i} \right. \\
&\quad - \left[-\frac{3xz}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \left(-\frac{3xz}{(x^2+y^2+z^2)^{\frac{5}{2}}} \right) \right] \vec{j} \\
&\quad \left. + \left[-\frac{3xy}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \left(-\frac{3xy}{(x^2+y^2+z^2)^{\frac{5}{2}}} \right) \right] \vec{k} \right\}
\end{aligned}$$

$$= \vec{0}$$

$\therefore \vec{E}$ is irrotational.

Summary of gradient, divergence and curl

- ◆ The vector differential operator ∇ :

$$\nabla \equiv \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$$

- ◆ Gradient of a scalar field f :

$$\text{grad } f = \nabla f$$

(the outcome is vector-valued)

- Divergence of a vector field $\vec{F}$:

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

(the outcome is scalar-valued)

- Curl of a vector field $\vec{F}$:

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

(the outcome is vector-valued)

- ◆ A vector field $\vec{v}$ is said to be solenoidal if $\text{div } \vec{v} = 0$.

A vector field $\vec{v}$ is said to be irrotational if $\text{curl } \vec{v} = \vec{0}$.

- ◆ Although divergence and gradient are defined in all dimensions, curl is defined only in three dimensions.

6 Vector Identities

Let f and g be scalar fields, $\vec{v}$ and $\vec{w}$ be vector fields, all are twice continuously differentiable.

Some useful results involving the operator ∇ are shown below:

$$(i) \quad \nabla(f + g) = \nabla f + \nabla g$$

$$(ii) \quad \nabla(fg) = f\nabla g + g\nabla f$$

$$(iii) \quad \nabla \cdot (\vec{v} + \vec{w}) = \nabla \cdot \vec{v} + \nabla \cdot \vec{w}$$

$$(iv) \quad \nabla \cdot (f\vec{v}) = (\nabla f) \cdot \vec{v} + f(\nabla \cdot \vec{v})$$

$$(v) \quad \nabla \times (\vec{v} + \vec{w}) = \nabla \times \vec{v} + \nabla \times \vec{w}$$

$$(vi) \quad \nabla \times (f\vec{v}) = (\nabla f) \times \vec{v} + f(\nabla \times \vec{v})$$

$$(vii) \quad \nabla \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\nabla \times \vec{v}) - \vec{v} \cdot (\nabla \times \vec{w})$$

$$(viii) \quad \nabla \times (\vec{v} \times \vec{w}) = (\vec{w} \cdot \nabla)\vec{v} - (\vec{v} \cdot \nabla)\vec{w} + (\nabla \cdot \vec{w})\vec{v} - (\nabla \cdot \vec{v})\vec{w}$$

$$\star (ix) \quad \nabla \times (\nabla f) = \vec{0} \quad (\text{i.e. } \text{curl}(\text{grad } f) = \vec{0})$$

$$\star (x) \quad \nabla \cdot (\nabla \times \vec{v}) = 0 \quad (\text{i.e. } \text{div}(\text{curl } \vec{v}) = 0)$$

$$(xi) \quad \nabla \cdot (\nabla f) = \nabla \cdot \left(\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

operator $w_1 \frac{\partial}{\partial x} + w_2 \frac{\partial}{\partial y} + w_3 \frac{\partial}{\partial z}$

scalar function
 $\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial z}$

Let $f(x, y, z)$ be a scalar field. Define the **Laplacian** $\nabla^2 f$ of f by

$$\nabla^2 f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

In ∇ notation,

$$\nabla^2 f = \nabla \cdot (\nabla f).$$

The operator

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

can be applied to a vector field

$$\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$$

and we get

$$\begin{aligned} \nabla^2 \vec{v} &\equiv \frac{\partial^2 \vec{v}}{\partial x^2} + \frac{\partial^2 \vec{v}}{\partial y^2} + \frac{\partial^2 \vec{v}}{\partial z^2} \\ &= \left(\frac{\partial^2 v_1}{\partial x^2} \vec{i} + \frac{\partial^2 v_2}{\partial x^2} \vec{j} + \frac{\partial^2 v_3}{\partial x^2} \vec{k} \right) + \left(\frac{\partial^2 v_1}{\partial y^2} \vec{i} + \frac{\partial^2 v_2}{\partial y^2} \vec{j} + \frac{\partial^2 v_3}{\partial y^2} \vec{k} \right) + \left(\frac{\partial^2 v_1}{\partial z^2} \vec{i} + \frac{\partial^2 v_2}{\partial z^2} \vec{j} + \frac{\partial^2 v_3}{\partial z^2} \vec{k} \right) \end{aligned}$$

Example 10

Prove (xi): $\nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$

Proof

Let f be a scalar field.

$$\begin{aligned}
 \nabla \cdot (\nabla f) &= \nabla \cdot \left(\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \right) \\
 &= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \right) \\
 &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial z} \right) \\
 &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f
 \end{aligned}$$

□

Example 11

Prove (vi): $\nabla \times (f\vec{v}) = (\nabla f) \times \vec{v} + f(\nabla \times \vec{v})$.

Proof

Let f be a scalar field and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be a vector field.

$$\begin{aligned}
 \nabla \times (f\vec{v}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fv_1 & fv_2 & fv_3 \end{vmatrix} \\
 &= \vec{i} \left[\frac{\partial}{\partial y} (fv_3) - \frac{\partial}{\partial z} (fv_2) \right] - \vec{j} \left[\frac{\partial}{\partial x} (fv_3) - \frac{\partial}{\partial z} (fv_1) \right] + \vec{k} \left[\frac{\partial}{\partial x} (fv_2) - \frac{\partial}{\partial y} (fv_1) \right] \\
 &\stackrel{\text{Product rule} \rightarrow}{=} \vec{i} \left[\frac{\partial f}{\partial y} v_3 + f \frac{\partial v_3}{\partial y} - \frac{\partial f}{\partial z} v_2 - f \frac{\partial v_2}{\partial z} \right] - \vec{j} \left[\frac{\partial f}{\partial x} v_3 + f \frac{\partial v_3}{\partial x} - \frac{\partial f}{\partial z} v_1 - f \frac{\partial v_1}{\partial z} \right] \\
 &\quad + \vec{k} \left[\frac{\partial f}{\partial x} v_2 + f \frac{\partial v_2}{\partial x} - \frac{\partial f}{\partial y} v_1 - f \frac{\partial v_1}{\partial y} \right] \\
 &= \vec{i} \left(\frac{\partial f}{\partial y} v_3 - \frac{\partial f}{\partial z} v_2 \right) + f \vec{i} \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) - \vec{j} \left(\frac{\partial f}{\partial x} v_3 - \frac{\partial f}{\partial z} v_1 \right) - f \vec{j} \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right) \\
 &\quad + \vec{k} \left(\frac{\partial f}{\partial x} v_2 - \frac{\partial f}{\partial y} v_1 \right) + f \vec{k} \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)
 \end{aligned}$$

$$\begin{aligned}
&= \underbrace{\vec{i} \left(\frac{\partial f}{\partial y} v_3 - \frac{\partial f}{\partial z} v_2 \right) - \vec{j} \left(\frac{\partial f}{\partial x} v_3 - \frac{\partial f}{\partial z} v_1 \right) + \vec{k} \left(\frac{\partial f}{\partial x} v_2 - \frac{\partial f}{\partial y} v_1 \right)}_{\text{green}} \\
&\quad + f \left[\vec{i} \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) - \vec{j} \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right) + \vec{k} \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \right] \\
&= \underbrace{\vec{i} \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ v_2 & v_3 \end{vmatrix} - \vec{j} \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \\ v_1 & v_3 \end{vmatrix} + \vec{k} \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ v_1 & v_2 \end{vmatrix}}_{\text{green}} \\
&\quad + f \left[\vec{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_2 & v_3 \end{vmatrix} - \vec{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ v_1 & v_3 \end{vmatrix} + \vec{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ v_1 & v_2 \end{vmatrix} \right]
\end{aligned}$$

$$\begin{aligned}
&\stackrel{\substack{\nabla f \\ \vec{v}}}{=} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} + \underbrace{f}_{\text{blue circle}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \quad \begin{matrix} \leftarrow \nabla \\ \leftarrow \vec{v} \end{matrix} \\
&= \underbrace{(\nabla f) \times \vec{v}}_{\text{red}} + \underbrace{f(\nabla \times \vec{v})}_{\text{green}}
\end{aligned}$$

$$\therefore \nabla \times (f\vec{v}) = (\nabla f) \times \vec{v} + f(\nabla \times \vec{v}).$$

□

Example 12

$$\text{curl}(\text{grad } f) = \vec{0}$$

Prove (ix): $\nabla \times (\nabla f) = \vec{0}$.

Proof

Let f be a twice continuously differentiable scalar field.

$$\begin{aligned} \nabla \times (\nabla f) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= \vec{i} \left[\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial y} \right) \right] - \vec{j} \left[\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial x} \right) \right] + \vec{k} \left[\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \right] \\ &= \vec{i} \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) - \vec{j} \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right) + \vec{k} \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \\ &= \vec{0} \end{aligned}$$

□

Now, let's consider result (ix) again:

 $\text{curl}(\text{grad } \varphi) = \nabla \times (\nabla \varphi) = \vec{0}.$

Therefore, **gradient fields are irrotational**. That is,

$$\vec{F} = \nabla \varphi \Rightarrow \text{curl } \vec{F} = \vec{0}.$$



The converse of this result is also true, that is, every irrotational vector field is the gradient of some scalar field.

That is, we have the result:



If $\vec{F}$ is irrotational (i.e. $\text{curl } \vec{F} = \vec{0}$),
then there exists a scalar field φ such that $\vec{F} = \nabla \varphi$.

Such scalar field φ is often called potential function.

This has important applications in electromagnetism and fluid dynamics.

Similarly, result (x) says that

$$\text{div}(\text{curl } \vec{v}) = \nabla \cdot (\nabla \times \vec{v}) = 0.$$

That is, **the curl of any vector field is solenoidal**.

$$\nabla \cdot \vec{F} = 0 \Rightarrow \vec{F} = \nabla \times \vec{v}$$

The converse of result (x), which says that **every solenoidal vector field is the curl of some vector field**, arises in applications.

That is, we have the result:



If $\vec{F}$ is solenoidal (i.e. $\text{div } \vec{F} = 0$),
then there exists some vector field $\vec{v}$ such that $\vec{F} = \text{curl } \vec{v}$.

Example 13

Show that the vector field

$$\vec{F}(x, y, z) = y \vec{i} + (x + z \cos y) \vec{j} + (\sin y + 2z) \vec{k}$$

is irrotational. Then find a potential function $\varphi(x, y, z)$ such that $\nabla\varphi = \vec{F}$.

Solution

$$\begin{aligned} \text{curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x + z \cos y & \sin y + 2z \end{vmatrix} \\ &= \vec{i} \left[\frac{\partial(\sin y + 2z)}{\partial y} - \frac{\partial(x + z \cos y)}{\partial z} \right] - \vec{j} \left[\frac{\partial(\sin y + 2z)}{\partial x} - \frac{\partial(y)}{\partial z} \right] \\ &\quad + \vec{k} \left[\frac{\partial(x + z \cos y)}{\partial x} - \frac{\partial(y)}{\partial y} \right] \\ &= \vec{i}(\cos y - \cos y) - \vec{j}(0 - 0) + \vec{k}(1 - 1) \\ &= \vec{0} \end{aligned}$$

$\therefore \vec{F}$ is irrotational.

Show th

$$\nabla \times \vec{F} = \vec{0}$$

refer to
p.49

(result (ix))

$$\nabla \times (\nabla \varphi) = \vec{0}$$

Then there exists a scalar function $\varphi(x, y, z)$ such that $\nabla\varphi = \vec{F}$, i.e.

$$\frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k} = y\vec{i} + (x + z\cos y)\vec{j} + (\sin y + 2z)\vec{k}$$

$$\Rightarrow \begin{cases} \frac{\partial\varphi}{\partial x} = y & \dots\dots (1) \\ \frac{\partial\varphi}{\partial y} = x + z\cos y & \dots\dots (2) \\ \frac{\partial\varphi}{\partial z} = \sin y + 2z & \dots\dots (3) \end{cases}$$

Integrating (1) w.r.t. x :

$$\varphi(x, y, z) = xy + g(y, z) \dots\dots (*)$$

where $g(y, z)$ is a function of y and z .

Partially differentiate both sides of (*) w.r.t. y :

$$\frac{\partial\varphi}{\partial y} = x + \frac{\partial g}{\partial y}$$

Comparing with (2):

$$\begin{aligned} \frac{\partial\varphi}{\partial x} &= y \\ \Rightarrow \varphi &= \int y dx = xy + g(y, z) \end{aligned}$$

$$x + \underbrace{z \cos y} = x + \underbrace{\frac{\partial g}{\partial y}} \Rightarrow \underbrace{\frac{\partial g}{\partial y}} = z \cos y.$$

Integrate w.r.t. y :

$$g(y, z) = z \sin y + h(z)$$

$$\begin{aligned} g(y, z) &= \int z \cos y \, dy \\ &= z \sin y + h(z) \end{aligned}$$

where $h(z)$ is a function of z .

$$\therefore \varphi(x, y, z) = \underline{xy} + \underline{z \sin y} + \underline{h(z)}.$$

Partially differentiate w.r.t. z :

$$\frac{\partial \varphi}{\partial z} = \sin y + h'(z)$$

Comparing with (3):

$$\sin y + 2z = \sin y + h'(z)$$

$$\Rightarrow h'(z) = 2z$$

$$\Rightarrow h(z) = z^2 + C, \text{ where } C \text{ is a constant.}$$

Hence,

$$\varphi(x, y, z) = xy + z \sin y + z^2 + C, \text{ where } C \text{ is a constant.}$$

Check:

$$\varphi(x, y, z) = xy + z \sin y + z^2 + C$$

Then

$$\begin{aligned}\nabla \varphi &= \frac{\partial \varphi}{\partial x} \vec{i} + \frac{\partial \varphi}{\partial y} \vec{j} + \frac{\partial \varphi}{\partial z} \vec{k} \\&= (y + 0 + 0 + 0) \vec{i} + (x + z \cos y + 0 + 0) \vec{j} + (0 + \sin y + 2z + 0) \vec{k} \\&= y \vec{i} + (x + z \cos y) \vec{j} + (\sin y + 2z) \vec{k} \\&= \vec{F} \quad \checkmark\end{aligned}$$

Example 14

result (x) on p. 43

$$\nabla \cdot (\nabla \times \vec{H}) = 0$$

$$\nabla \cdot \vec{F} = 0$$

Let $\vec{F} = (y - z)\vec{i} + (z - x)\vec{j} + (x - y)\vec{k}$. Show that $\vec{F}$ is solenoidal. Then find a vector field $\vec{H} = h_1(x, y, z)\vec{i} + h_2(x, y, z)\vec{j} + h_3(x, y, z)\vec{k}$ with $h_1 \equiv 0$ such that $\vec{F} = \nabla \times \vec{H}$.

Solution

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(y - z) + \frac{\partial}{\partial y}(z - x) + \frac{\partial}{\partial z}(x - y) = 0 + 0 + 0 = 0.$$

Hence, $\vec{F}$ is solenoidal.

Then there exists a vector field $\vec{H} = h_1(x, y, z)\vec{i} + h_2(x, y, z)\vec{j} + h_3(x, y, z)\vec{k}$ such that $\vec{F} = \nabla \times \vec{H}$. We try to find the vector $\vec{H}$ with $h_1 \equiv 0$. Then

$$\nabla \times \vec{H} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & h_2 & h_3 \end{pmatrix} = \left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} \right) \vec{i} - \frac{\partial h_3}{\partial x} \vec{j} + \frac{\partial h_2}{\partial x} \vec{k}$$

$$\vec{F} = \nabla \times \vec{H} \Rightarrow (y - z)\vec{i} + (z - x)\vec{j} + (x - y)\vec{k} = \left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} \right) \vec{i} - \frac{\partial h_3}{\partial x} \vec{j} + \frac{\partial h_2}{\partial x} \vec{k}$$

That is,

$$\begin{cases} y - z = \frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} & \dots (1) \\ z - x = -\frac{\partial h_3}{\partial x} & \dots (2) \\ x - y = \frac{\partial h_2}{\partial x} & \dots (3) \end{cases}$$

From (3):

$$\frac{\partial h_2}{\partial x} = x - y \Rightarrow h_2 = \int (x - y) dx = \frac{x^2}{2} - xy + \underline{A(y, z)},$$

where $A(y, z)$ is a function of y and z .

Set $A(y, z) = 0$. Then

$$h_2 = \frac{x^2}{2} - xy.$$

a function of the
2 variables
that have
been treated
as constants

From (2):

$$\frac{\partial h_3}{\partial x} = x - z \Rightarrow h_3 = \int (x - z) dx = \frac{x^2}{2} - xz + B(y, z),$$

where $B(y, z)$ is a function of y and z .

From (1):

$$y - z = \frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} = \frac{\partial}{\partial y} \left[\frac{x^2}{2} - xz + B(y, z) \right] - \frac{\partial}{\partial z} \left[\frac{x^2}{2} - xy \right] = \frac{\partial B(y, z)}{\partial y}$$

Integrating both sides w.r.t. y gives

$$B(y, z) = \int (y - z) dy = \frac{y^2}{2} - yz + C(z).$$

Set $C(z) = 0$, then $B(y, z) = \frac{y^2}{2} - yz$.

Thus,

$$h_3 = \frac{x^2}{2} - xz + B(y, z) = \frac{x^2}{2} + \frac{y^2}{2} - xz - yz.$$

Hence,

$$\vec{H} = \left(\frac{x^2}{2} - xy \right) \vec{j} + \left(\frac{x^2}{2} + \frac{y^2}{2} - xz - yz \right) \vec{k}$$

(Notice that $\vec{H}$ is not unique.)