

L08: Introduction to Transmission Lines

EE3109 Applied Electromagnetics

Lecture Notes

Dr. Alex M. H. Wong

Jan – Apr 2023

Outline

- Course Matter
- Transmission Lines – Bridging electromagnetics with circuit theory
- The TL Model
- Basic TL Relationships

Announcements

- Assignment 2 due today
- Laboratory Starts This Week
 - Please **keep to your registered laboratory session**
 - Please arrive at the laboratory **on time**: it will facilitate your groupings and timely completion of the lab
 - Please stay with your **lab partner(s) as much as possible** throughout both labs
- Meta-Devices Workshop – tomorrow!

Tentative Course Schedule

Week #	Week of	Course Coverage
1	Jan 9	Course Introduction, Math Review
2	Jan 16	Introduction to Maxwell Equations, Materials and Boundaries
	Jan 23	<i>Happy Chinese New Year!</i>
3	Jan 30	Wave Equations, Plane Waves, Propagation in Lossy Media
4	Feb 6	Polarizations, <i>Asst. 1a Due</i>
5	Feb 13	EM Potential, Polarizations Cont'd, Potentials <i>Asst. 1b Due</i>
6	Feb 20	<i>Test 1</i> , Reflection and Transmission
7	Feb 27	Reflection and Transmission Cont'd
8	Mar 6	Power and Velocity
9	Mar 13	Waveguides and Transmission Lines (TL), VIP in a TL, Terminated and Lossy TLs, <i>Asst. 2 Due, Lab 1 Begins</i>
10	Mar 20	Impedance in a TL, Smith Charts
11	Mar 27	Smith Charts Cont'd, Impedance Matching, Quarter-wave Transformers, <i>Lab 2 Begins</i>
12	Apr 3	<i>Test 2</i> , Introduction to Antennas
13	Apr 10	Dipole Antennas, <i>Asst. 3 Due</i>

Outline

- Course Matter
- Transmission Lines – Bridging electromagnetics with circuit theory
- The TL Model
- Basic TL Relationships

What is a Transmission Line?

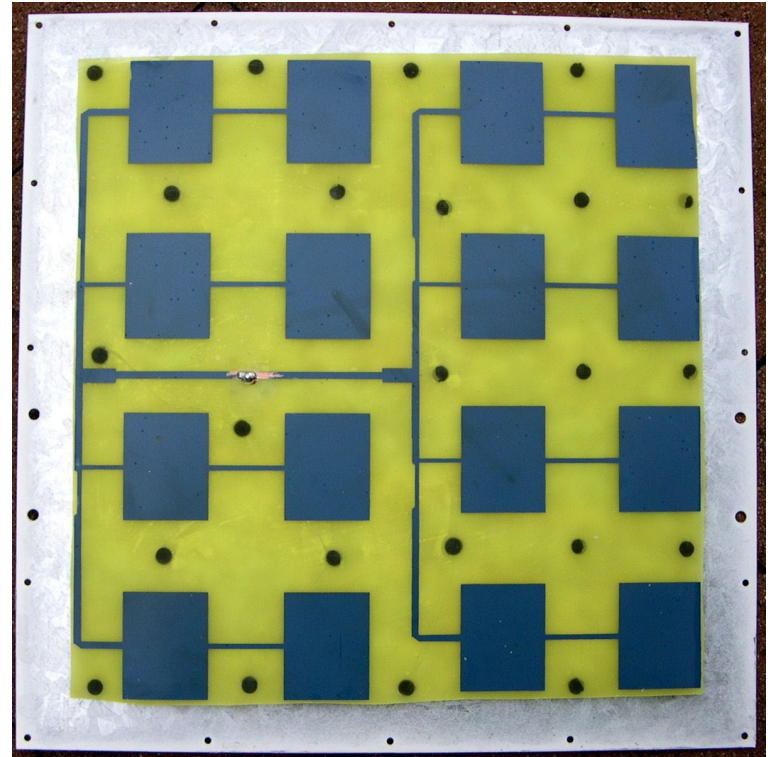


<https://i.stack.imgur.com/H5vVZ.jpg>

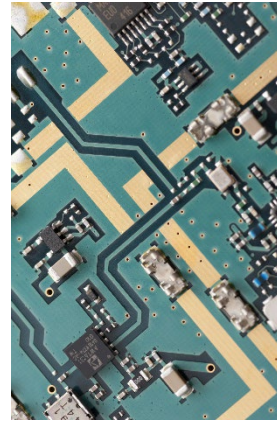
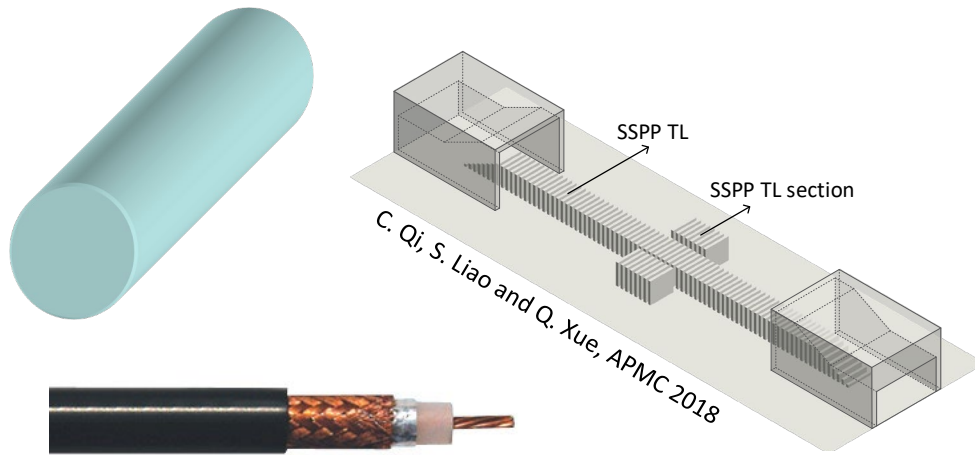
What is a Transmission Line?



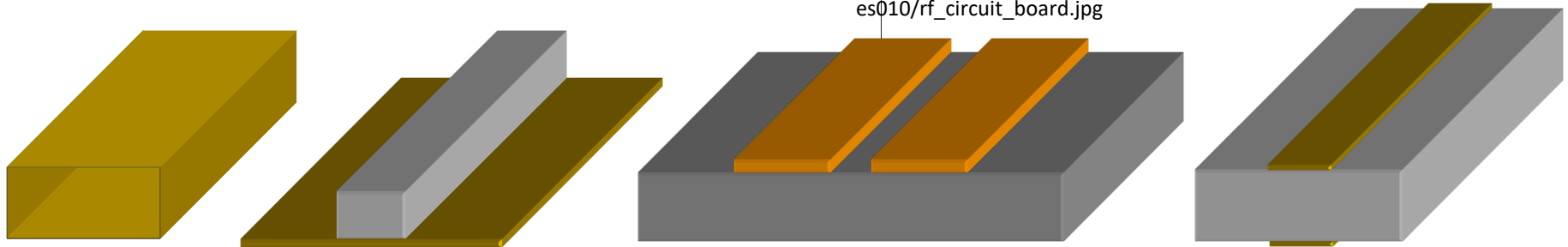
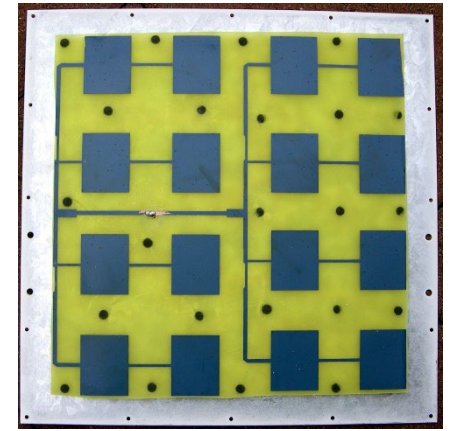
http://farm3.staticflickr.com/2035/2308155830_a7c281a31a_z.jpg?zz=1



https://upload.wikimedia.org/wikipedia/commons/thumb/1/16/Antenne_patch_2.4_GHz.JPG/1200px-Antenne_patch_2.4_GHz.JPG



http://www.freeimageslive.co.uk/files/images010/rf_circuit_board.jpg



A **transmission line (TL)** is a structure that guides EM waves in a desired direction. Usually the length of the TL is appreciable compared to the wavelength, such that a change in phase can be observed in the voltage or current wave.

Example: the Optical Fiber

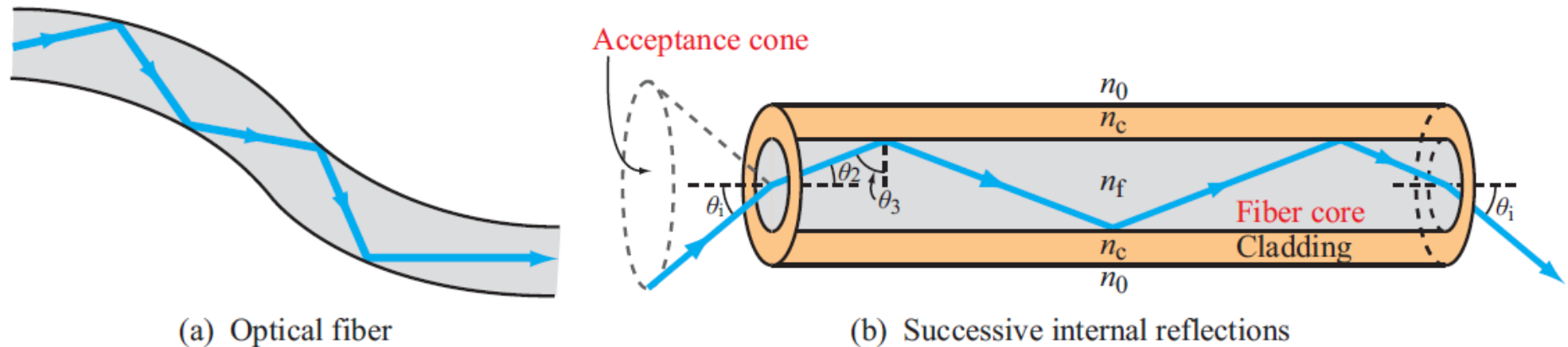


Figure 8-12: Waves can be guided along optical fibers as long as the reflection angles exceed the critical angle for total internal reflection.

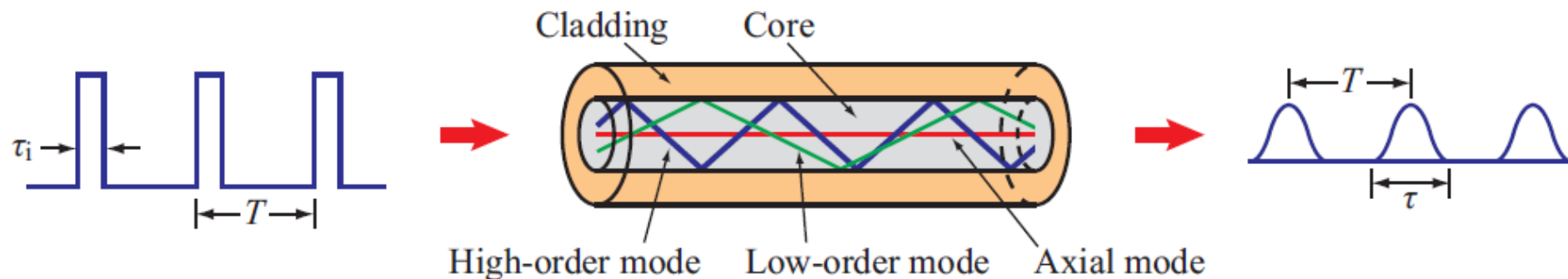
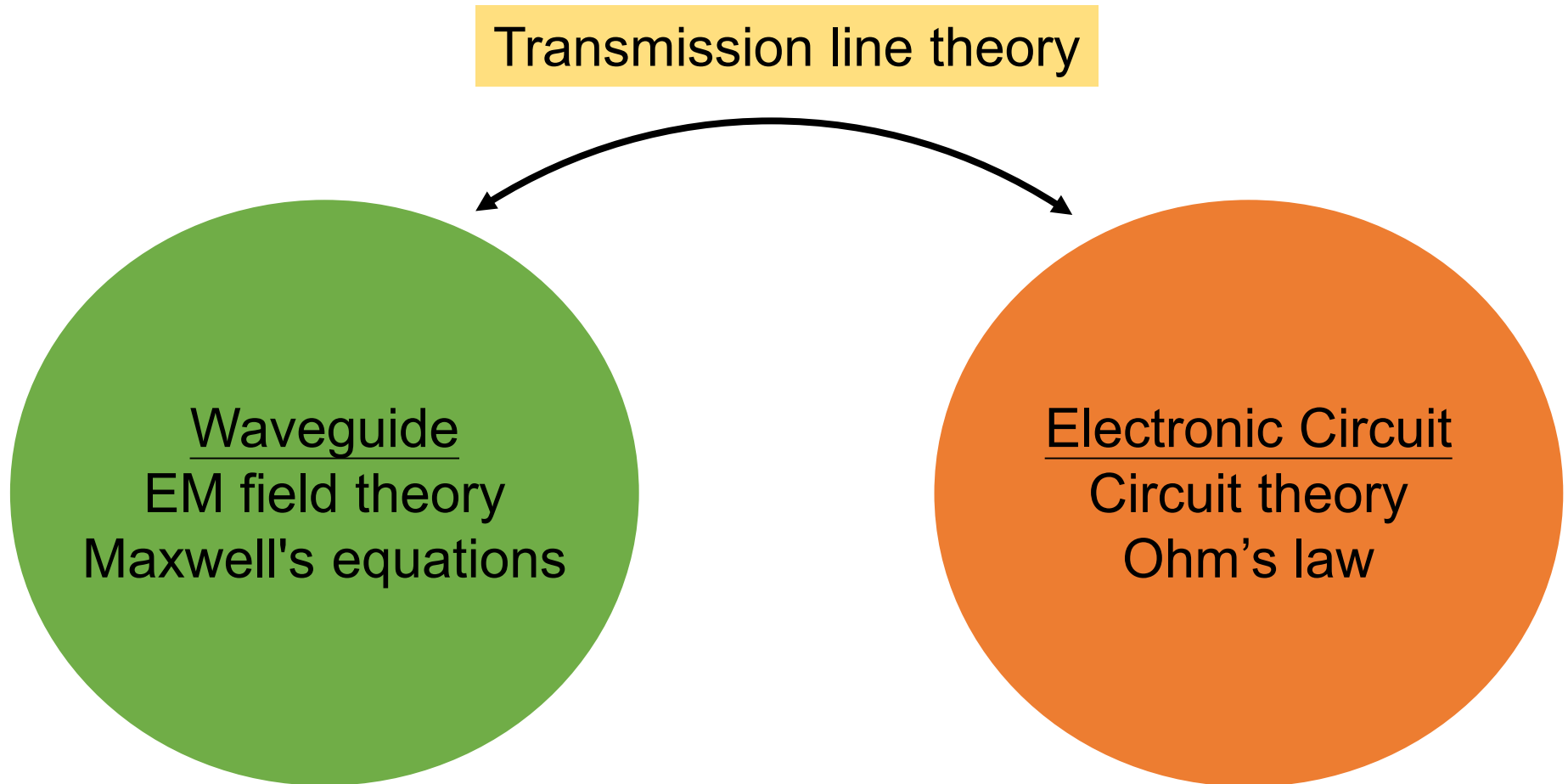


Figure 8-13: Distortion of rectangular pulses caused by modal dispersion in optical fibers.

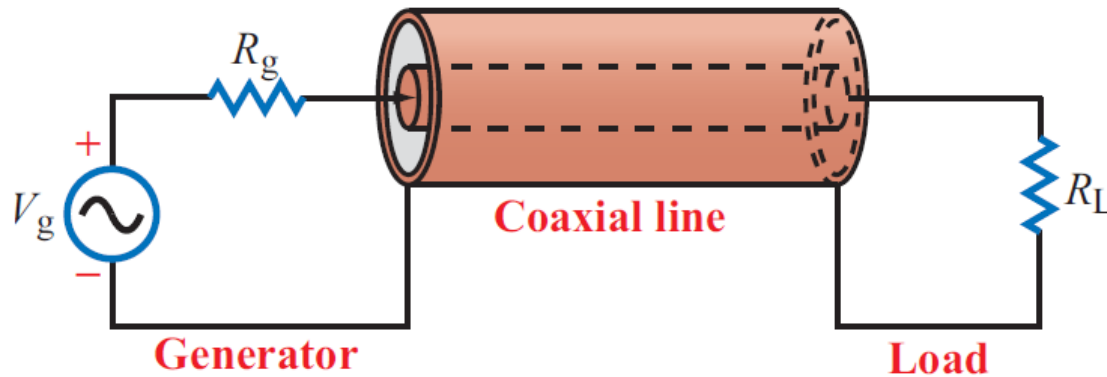
Waveguides & Transmission Lines

- **Waveguide**: general EM structure that guide waves
 - Has **many different “modes”**: orthogonal E-field and H-field distributions along the cross-section, that don't change in the longitudinal direction
- **Transmission Lines (TL)**: abstracted single mode waveguides
 - When a waveguide has **one mode**, one can simplify its analysis defining currents, voltages and resistances on the line, thereby **simplifying the EM problem to a circuit problem**

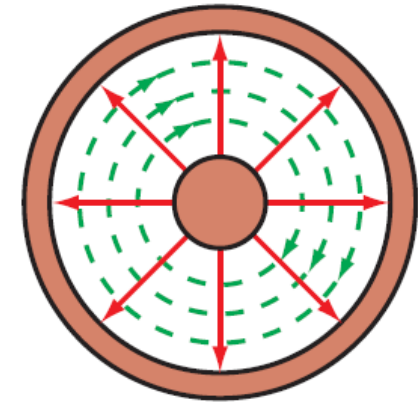
Transmission Lines vs. Waveguides



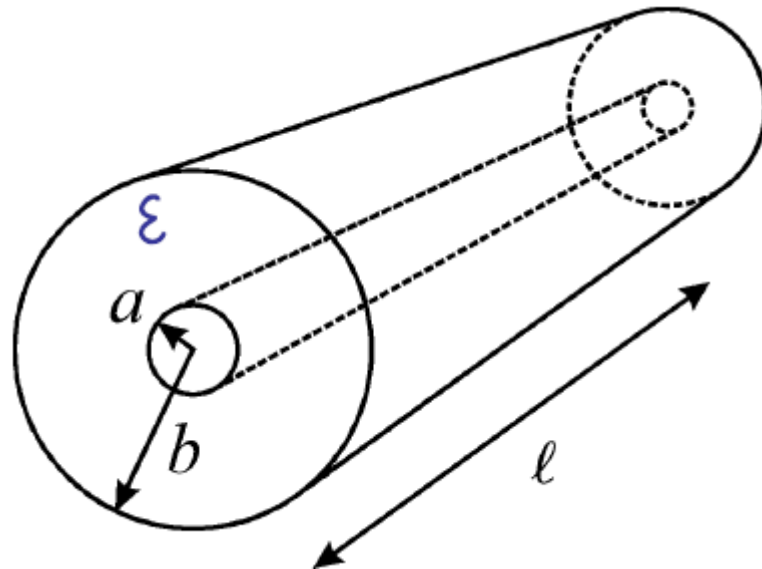
Example: Coaxial Cable



-- Magnetic field lines
— Electric field lines

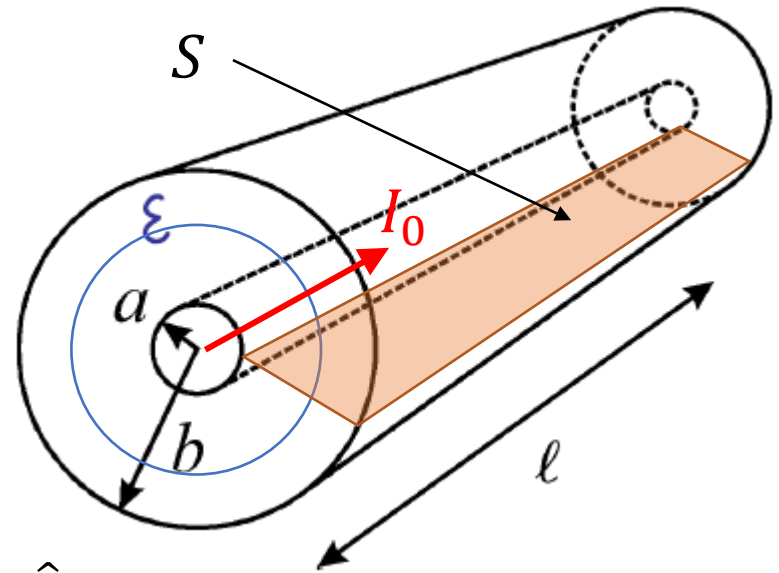


Cross section



Electric Field E is radial
Magnetic Field H is azimuthal
Propagation is into the page

Inductance



For $a < r < b$,

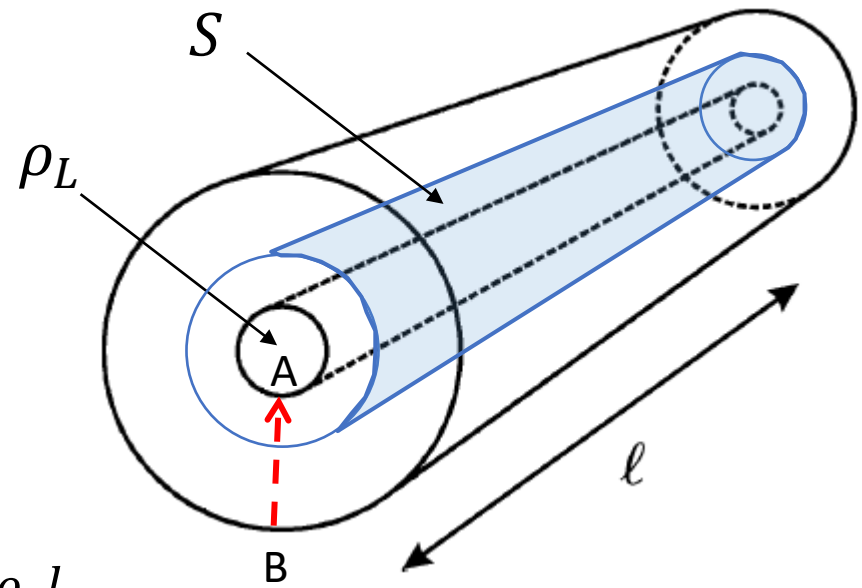
$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_0 = H_\phi 2\pi r = I_0 \Rightarrow \mathbf{H} = \frac{I_0}{2\pi r} \hat{\mathbf{a}}_\phi$$

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I_0}{2\pi r} \hat{\mathbf{a}}_\phi$$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_0^\ell \int_a^b \frac{\mu_0 I_0}{2\pi r} dr dz = \frac{\mu_0 I_0 \ell}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$L = \frac{\Phi}{I_0} = \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{b}{a}\right)$$

Capacitance



For $a < r < b$,

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{enc} \Rightarrow D_\rho 2\pi r l = \rho_L l$$

$$D_\rho = \frac{\rho_L}{2\pi r}; E_\rho = \frac{D_\rho}{\epsilon} = \frac{\rho_L}{2\pi r \epsilon}$$

$$V = - \int_B^A \mathbf{E} \cdot d\mathbf{l} = - \int_b^a E_\rho dr = \frac{\rho_L}{2\pi \epsilon} \ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q_{enc}}{V} = 2\pi \epsilon l \left[\ln\left(\frac{b}{a}\right) \right]^{-1}$$

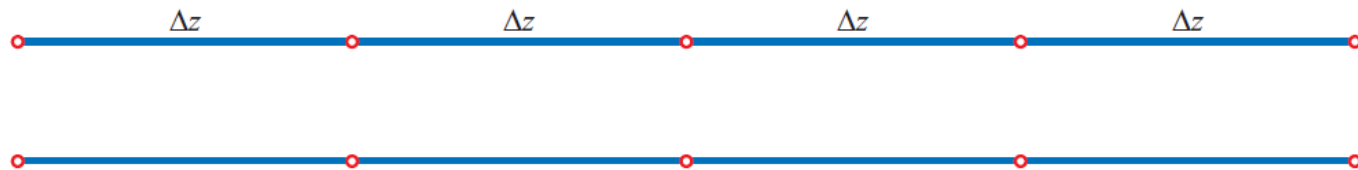
Outline

- Course Matter
- Transmission Lines – Bridging electromagnetics with circuit theory
- The TL Model
- Basic TL Relationships

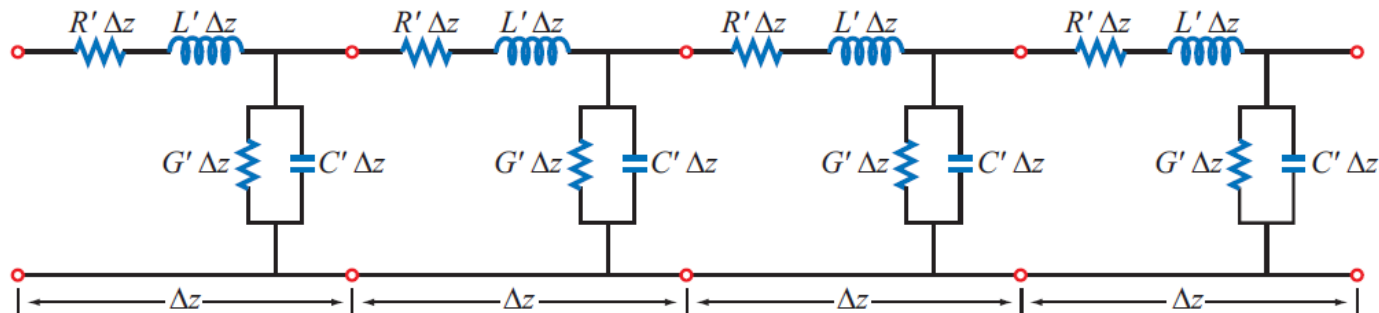
Transmission Line Model



(a) Parallel-wire representation



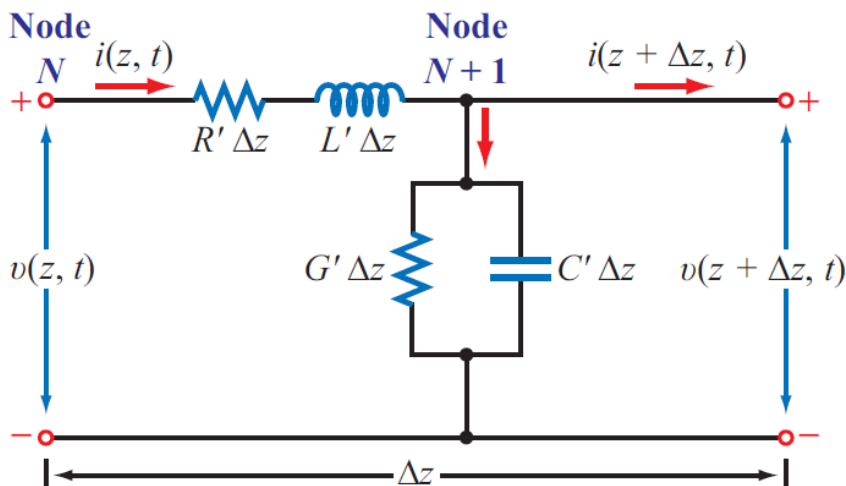
(b) Differential sections each Δz long



(c) Each section is represented by an equivalent circuit

- R' : The combined **resistance** of both conductors per unit length, in Ω/m ,
- L' : The combined **inductance** of both conductors per unit length, in H/m ,
- G' : The **conductance** of the insulation medium between the two conductors per unit length, in S/m , and
- C' : The **capacitance** of the two conductors per unit length, in F/m .

Transmission-Line Equations



$$i(z, t) - G' \Delta z v(z + \Delta z, t) - C' \Delta z \frac{\partial v(z + \Delta z, t)}{\partial t} - i(z + \Delta z, t) = 0. \quad (2.15)$$

Upon dividing all terms by Δz and taking the limit $\Delta z \rightarrow 0$, Eq. (2.15) becomes a second-order differential equation:

$$v(z, t) - R' \Delta z i(z, t) - L' \Delta z \frac{\partial i(z, t)}{\partial t} - v(z + \Delta z, t) = 0.$$

$$-\frac{\partial i(z, t)}{\partial z} = G' v(z, t) + C' \frac{\partial v(z, t)}{\partial t}. \quad (2.16)$$

Upon dividing all terms by Δz and rearranging them, we obtain

$$-\left[\frac{v(z + \Delta z, t) - v(z, t)}{\Delta z} \right] = R' i(z, t) + L' \frac{\partial i(z, t)}{\partial t}. \quad (2.13)$$

In the limit as $\Delta z \rightarrow 0$, Eq. (2.13) becomes a differential equation:

$$-\frac{\partial v(z, t)}{\partial z} = R' i(z, t) + L' \frac{\partial i(z, t)}{\partial t}. \quad (2.14)$$

AC signals: use phasors!

$$v(z, t) = \Re[\tilde{V}(z) e^{j\omega t}],$$

$$i(z, t) = \Re[\tilde{I}(z) e^{j\omega t}],$$

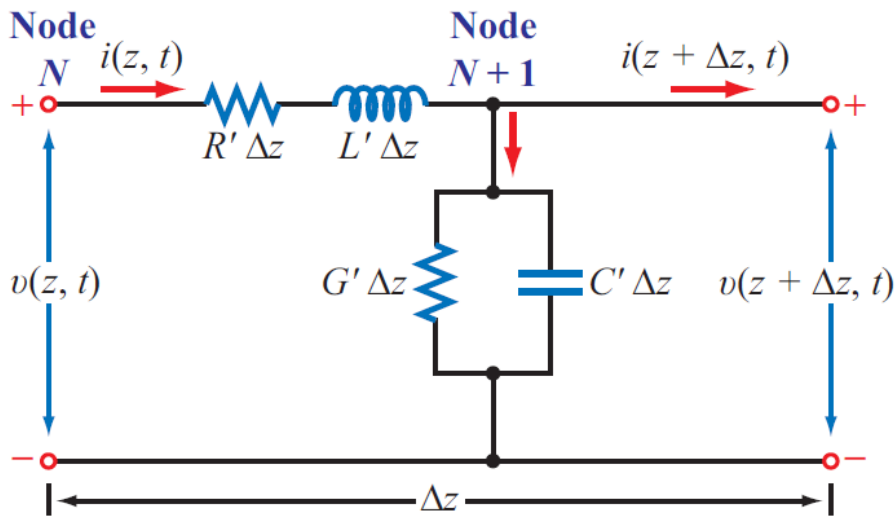
$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L') \tilde{I}(z),$$

$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C') \tilde{V}(z).$$

Telegrapher's equations

Recap: Transmission Line Model

Solving this electronic circuit gives:



$$\begin{aligned} -\frac{dV(z)}{dz} &= (R' + j\omega L')I(z) ; \\ -\frac{dI(z)}{dz} &= (G' + j\omega C')V(z) . \end{aligned}$$

(Telegrapher's Equations)

(Can we get uncoupled equations... one for $V(z)$ and one for $I(z)$?)

Wave Equations for TL

Telegrapher's Equations

$$-\frac{dV(z)}{dz} = (R' + j\omega L')I(z); \quad (1)$$

$$-\frac{dI(z)}{dz} = (G' + j\omega C')V(z). \quad (2)$$

Take d/dz of (1) and sub
in (2);
Take d/dz of (2) and sub
in (1).

Wave Equations

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0; \quad (3)$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0. \quad (4)$$

where $\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')} = \alpha + j\beta$
($\gamma \equiv$ propagation constant; $\alpha \equiv$ attenuation
constant; $\beta \equiv$ phase constant)

Solutions to Wave Equations

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}; \quad (5)$$

$$I(z) = I_0^+ e^{-\gamma z} - I_0^- e^{+\gamma z}; \quad (6)$$

$$I(z) = \frac{1}{Z_0} (V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}), \text{ where}$$

Rearrange (6) to get:

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \equiv \text{Characteristic Impedance}$$

Outline

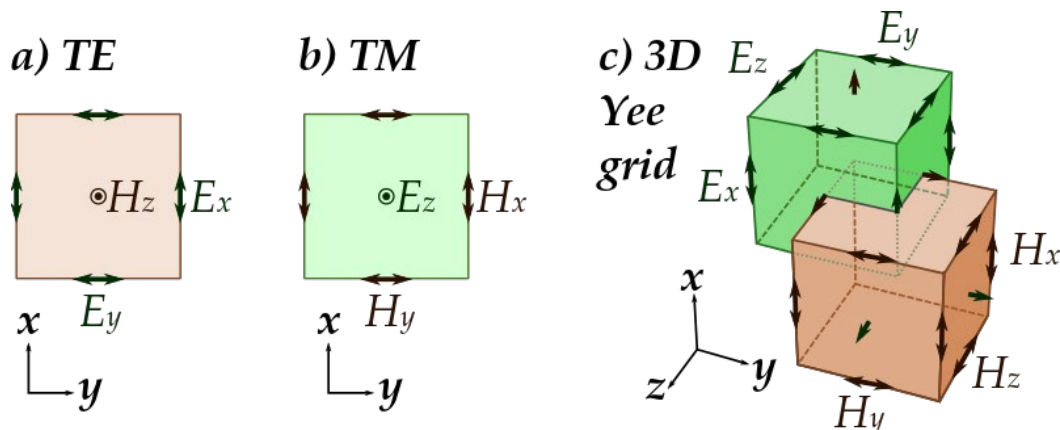
- Course Matter
- Transmission Lines – Bridging electromagnetics with circuit theory
- The TL Model
- Basic TL Relationships

Comparison: EM vs. TL

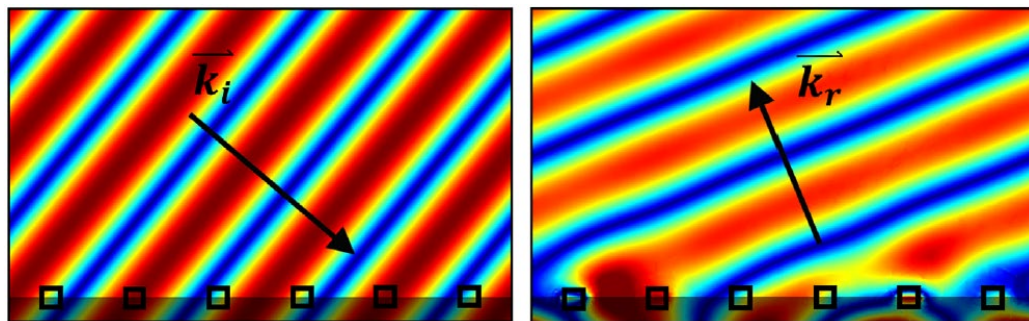
Electromagnetics	Transmission Lines
$\mathbf{E} ; \mathbf{H}$	$V = \int_a^b \mathbf{E} \cdot d\mathbf{l} ; I = \oint_c \mathbf{H} \cdot d\mathbf{l}$
Maxwell's Equations: $\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} ; \nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$	Telegrapher's Equations: $-\frac{dV}{dz} = (R' + j\omega L')I ; -\frac{dI}{dz} = (G' + j\omega C')V$
Wave Equations: $\frac{d^2 \mathbf{E}}{dz^2} - \gamma^2 \mathbf{E} = 0 ; \frac{d^2 \mathbf{H}}{dz^2} - \gamma^2 \mathbf{H} = 0$	Wave Equations: $\frac{d^2 V}{dz^2} - \gamma^2 V = 0 ; \frac{d^2 I}{dz^2} - \gamma^2 I = 0$
Solution to Wave Equations: $\mathbf{E}(\mathbf{r}) = (E_0^+ e^{-j\mathbf{k} \cdot \mathbf{r}} + E_0^- e^{+j\mathbf{k} \cdot \mathbf{r}}) \hat{\mathbf{e}}$ $\mathbf{H}(\mathbf{r}) = (H_0^+ e^{-j\mathbf{k} \cdot \mathbf{r}} - H_0^- e^{+j\mathbf{k} \cdot \mathbf{r}}) \hat{\mathbf{h}}$	Solution to Wave Equations: $V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} ;$ $I(z) = I_0^+ e^{-\gamma z} - I_0^- e^{+\gamma z}$
Intrinsic Impedance: $\eta = \frac{E_0^+}{H_0^+} = \sqrt{\frac{\mu}{\varepsilon_c}}$	Characteristic Impedance: $Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$
Propagation Constant: $jk = \gamma = j\omega \sqrt{\mu \varepsilon_c} = \alpha + j\beta$	Propagation Constant: $\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')} = \alpha + j\beta$

Computational Electromagnetics:

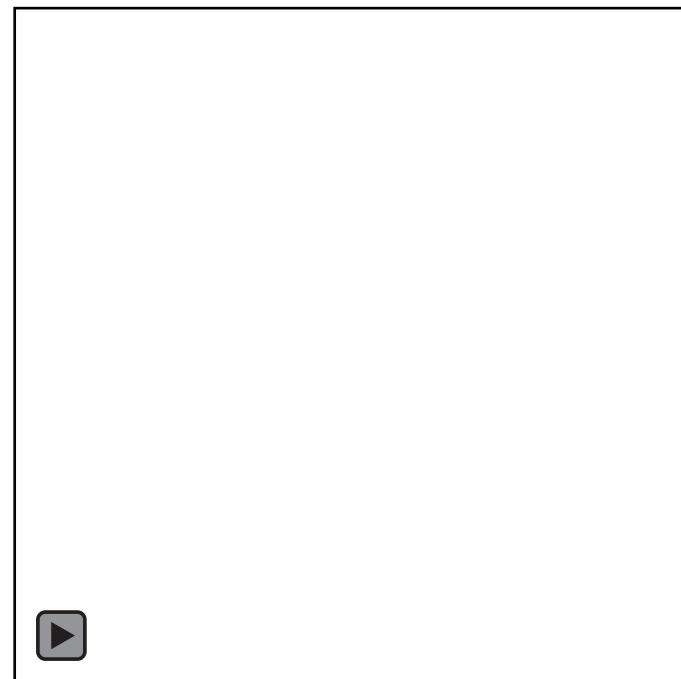
Finite Difference Time Domain (FDTD) and
Finite Element Method (FEM)



https://commons.wikimedia.org/wiki/File:FDTD_Yee_grid_2d-3d.svg



AMH Wong and GV Eleftheriades, Phys. Rev. X., 2018.

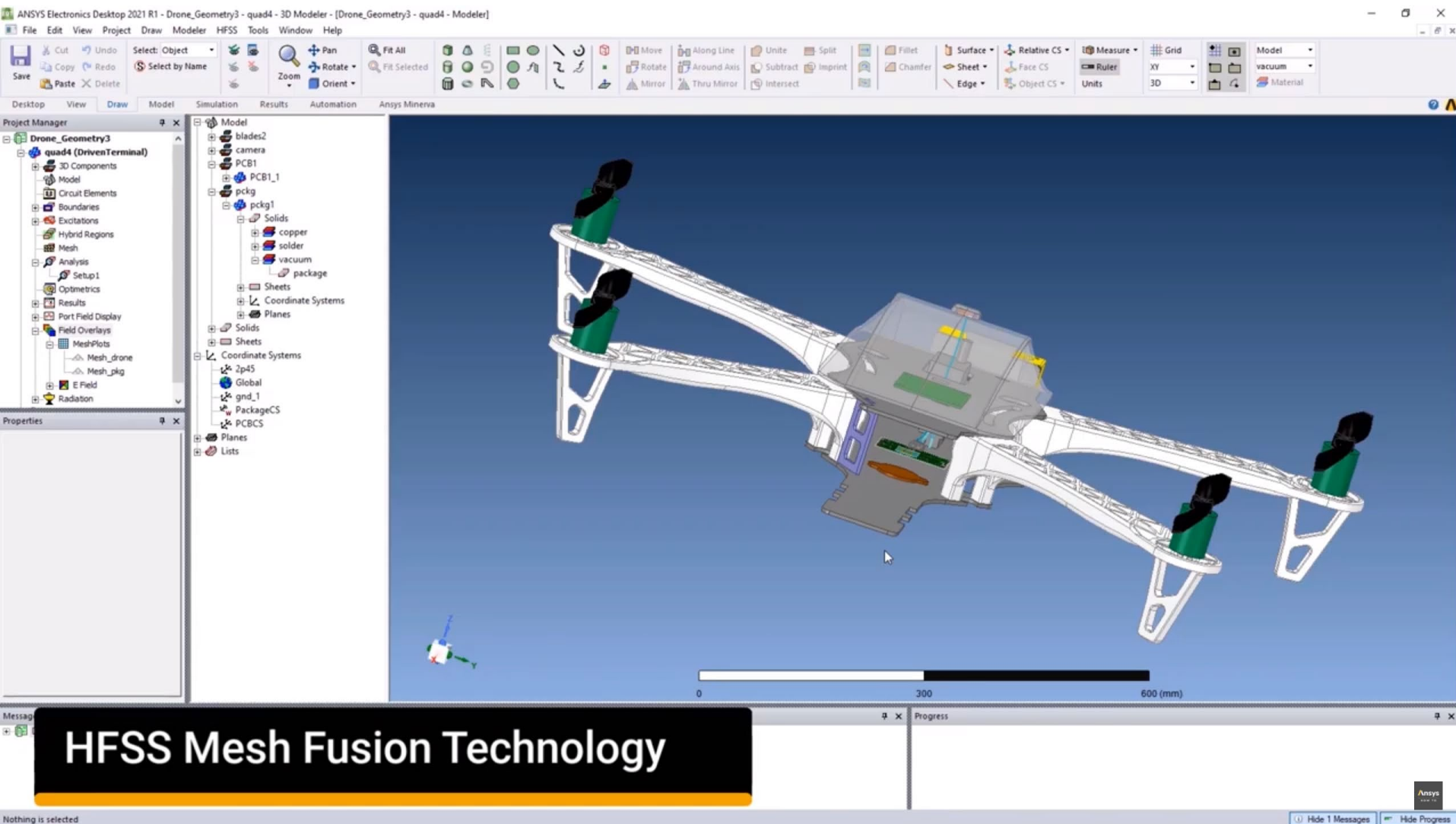


AMH Wong et al., Trans. Antennas Propag., 2021.

- **3D space** is treated as a **3D TL circuit** in a time-domain (FDTD) or frequency domain (FEM) simulation

Computational Electromagnetics

Eg. Ansys HFSS (You will explore this in Lab 1!)



Travelling Wave in a TL

Consider a positively travelling wave (in a lowloss TL):

$$V^+(z) = V_0^+ e^{-\alpha z} e^{-j\beta z} \Rightarrow V(z, t) = |V_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \angle V_0^+)$$

$$I^+(z) = \frac{V_0^+}{Z_0} e^{-\alpha z} e^{-j\beta z} \Rightarrow I(z, t) = \frac{|V_0^+|}{Z_0} e^{-\alpha z} \cos(\omega t - \beta z + \angle V_0^+)$$

Voltage amplitudes:

$$V_{p-p} = 2|V_0^+|; V_{RMS} = \frac{|V_0^+|}{\sqrt{2}}.$$

Consider the phase of this travelling wave ($\varphi = \omega t - \beta z + \angle V_0^+$):

The wavelength (λ) is the distance through which the wave changes in phase by 2π rad:

$$\lambda \equiv |\Delta z| \text{ s.t. } \Delta\varphi = 2\pi \Rightarrow \lambda = \frac{2\pi}{\beta}$$

The phase velocity (v_p) of the wave is the rate at which the constant phase point travels:

$$\text{Rearrange the phase expression to get } z = \frac{\omega}{\beta} t - \text{Constant} \Rightarrow v_p = \frac{dz}{dt} = \frac{\omega}{\beta}$$

Power in a TL

Consider a positively travelling wave (in a lowloss TL):

$$V^+(z) = V_0^+ e^{-\alpha z} e^{-j\beta z} \Rightarrow V(z, t) = |V_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \angle V_0^+)$$

$$I^+(z) = \frac{V_0^+}{Z_0} e^{-\alpha z} e^{-j\beta z} \Rightarrow I(z, t) = \frac{|V_0^+|}{Z_0} e^{-\alpha z} \cos(\omega t - \beta z + \angle V_0^+)$$

The **instantaneous power** carried is:

$$P(z, t) = V(z, t)I(z, t) = \frac{|V_0^+|^2}{Z_0} e^{-2\alpha z} \cos^2(\omega t - \beta z + \angle V_0^+)$$

The **time-average power** carried is:

$$P_{av}(z) = \frac{1}{T} \int_0^T P(z, t) dt = \frac{|V_0^+|^2}{Z_0} e^{-2\alpha z} \left(\frac{1}{T} \right) \int_0^T \cos^2(\omega t - \beta z + \angle V_0^+) dt$$

$$= \frac{|V_0^+|^2}{Z_0 T} e^{-2\alpha z} \int_0^T \frac{1 + \cos(2(\omega t - \beta z + \angle V_0^+))}{2} dt$$

$$= \frac{|V_0^+|^2}{2Z_0 T} e^{-2\alpha z} \left[t + \frac{1}{2\omega} \cos(2(\omega t - \beta z + \angle V_0^+)) \right]_{t=0}^{t=T}$$

$T = \frac{2\pi}{\omega}$ is the period
of $V(z, t)$ and $I(z, t)$

This term takes on the
same value for $t = 0$
and $t = T$.

$$P_{av}(z) = \frac{|V_0^+|^2}{2Z_0} e^{-2\alpha z} = \frac{1}{2} \operatorname{Re}\{V(z)I(z)^*\}$$

More on Impedance

$$\text{Characteristic Impedance: } Z_0 = \frac{V_0^+}{I_0^+} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$$

- Characteristic Impedance is a property of the frequency and the transmission line
 - In general, the characteristic impedance is a function of frequency
 - However, this will not be the case for lossless TLs.
- Impedance (looking toward the load) is given by:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}}{V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}}$$

Lossless TLs

- Condition: $R' = 0, G' = 0$ (reasonable approximation for many TLs)
- Consequences:

$$\gamma = \sqrt{(j\omega L')(j\omega C')} = j\omega\sqrt{L'C'}$$

$$\Rightarrow \alpha = 0 ; \beta = \omega\sqrt{L'C'}$$

$$Z_0 = \sqrt{\frac{j\omega L'}{j\omega C'}} = \sqrt{\frac{L'}{C'}}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$$

Lowloss TLs

- Condition: $R' \ll \omega L', G' \ll \omega C'$
- Consequences: (you will derive this in a future assignment)

$$\gamma = j\omega\sqrt{L'C'} + \frac{1}{2}\left(\frac{R'}{Z_0} + G'Z_0\right)$$

$$\Rightarrow \alpha = \frac{1}{2}\left(\frac{R'}{Z_0} + G'Z_0\right); \beta = \omega\sqrt{L'C'}$$

$$Z_0 = \sqrt{\frac{L'}{C'}}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$$

Example 1: Wire or TL?

2.2 A transmission line of length l connects a load to a sinusoidal voltage source with an oscillation frequency f . Assuming the velocity of wave propagation on the line is c , for which of the following situations is it reasonable to ignore the presence of the transmission line in the solution of the circuit:

- (a) $l = 30$ cm, $f = 20$ kHz,
- (b) $l = 50$ km, $f = 60$ Hz,
- (c) $l = 30$ cm, $f = 600$ MHz,
- (d) $l = 2$ mm, $f = 100$ GHz.

Solution: A transmission line is negligible when $l/\lambda \leq 0.01$.

- (a) $\frac{l}{\lambda} = \frac{lf}{u_p} = \frac{(30 \times 10^{-2} \text{ m}) \times (20 \times 10^3 \text{ Hz})}{3 \times 10^8 \text{ m/s}} = 2 \times 10^{-5}$ (negligible).
- (b) $\frac{l}{\lambda} = \frac{lf}{u_p} = \frac{(50 \times 10^3 \text{ m}) \times (60 \times 10^0 \text{ Hz})}{3 \times 10^8 \text{ m/s}} = 0.01$ (borderline).
- (c) $\frac{l}{\lambda} = \frac{lf}{u_p} = \frac{(30 \times 10^{-2} \text{ m}) \times (600 \times 10^6 \text{ Hz})}{3 \times 10^8 \text{ m/s}} = 0.60$ (nonnegligible).
- (d) $\frac{l}{\lambda} = \frac{lf}{u_p} = \frac{(2 \times 10^{-3} \text{ m}) \times (100 \times 10^9 \text{ Hz})}{3 \times 10^8 \text{ m/s}} = 0.66$ (nonnegligible).

Example 2: Air Line

An *air line* is a transmission line in which air separates the two conductors, which renders $G' = 0$ because $\sigma = 0$. In addition, assume that the conductors are made of a material with high conductivity so that $R' \simeq 0$. For an air line with a characteristic impedance of $50 \, \Omega$ and a phase constant of $20 \, \text{rad/m}$ at $700 \, \text{MHz}$, find the line inductance L' and the line capacitance C' .

Solution: The following quantities are given:

$$Z_0 = 50 \, \Omega, \quad \beta = 20 \, \text{rad/m}, \\ f = 700 \, \text{MHz} = 7 \times 10^8 \, \text{Hz}.$$

$$\begin{aligned} C' &= \frac{\beta}{\omega Z_0} \\ &= \frac{20}{2\pi \times 7 \times 10^8 \times 50} \\ &= 9.09 \times 10^{-11} \, (\text{F/m}) = 90.9 \, (\text{pF/m}). \end{aligned} \qquad \begin{aligned} L' &= Z_0^2 C' \\ &= (50)^2 \times 90.9 \times 10^{-12} \\ &= 2.27 \times 10^{-7} \, (\text{H/m}) = 227 \, (\text{nH/m}). \end{aligned}$$

Example 3: Distortionless Line

2.13 In addition to not dissipating power, a lossless line has two important features: (1) it is dispersionless (μ_p is independent of frequency) and (2) its characteristic impedance Z_0 is purely real. Sometimes, it is not possible to design a transmission line such that $R' \ll \omega L'$ and $G' \ll \omega C'$, but it is possible to choose the dimensions of the line and its material properties so as to satisfy the condition

$$R'C' = L'G' \quad (\text{distortionless line}).$$

Such a line is called a *distortionless* line because despite the fact that it is not lossless, it does nonetheless possess the previously mentioned features of the loss line. Show that for a distortionless line,

$$\alpha = R' \sqrt{\frac{C'}{L'}} = \sqrt{R'G'}, \quad \beta = \omega \sqrt{L'C'}, \quad Z_0 = \sqrt{\frac{L'}{C'}}.$$

(Please try for yourselves!)

Summary

- In this class, we
 - Introduced the transmission line model
 - Derived the Telegrapher's Equations and the waves equations
 - Introduced some basic TL Relationships
 - Introduced the lossless TL