

EE3109 Applied Electromagnetics
2022/23 Semester B
Week 5-7 Tutorial Solution

Q1. Sol'n.

8.12 A plane wave of unknown frequency is normally incident in air upon the surface of a perfect conductor. Using an electric-field meter, it was determined that the total electric field in the air medium is always zero when measured at a distance of 3 m from the conductor surface. Moreover, no such nulls were observed at distances closer to the conductor. What is the frequency of the incident wave?

Solution: The electric field of the standing wave is zero at the conductor surface, and the standing wave pattern repeats itself every $\lambda/2$. Hence,

$$\frac{\lambda}{2} = 3 \text{ m}, \quad \text{or } \lambda = 6 \text{ m},$$

in which case

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8}{6} = 5 \times 10^7 = 50 \text{ MHz}.$$

Q2. Sol'n.

8.13 Orange light of wavelength $0.61 \mu\text{m}$ in air enters a block of glass with $\epsilon_r = 1.44$. What color would it appear to a sensor embedded in the glass? The wavelength ranges of colors are violet (0.39 to $0.45 \mu\text{m}$), blue (0.45 to $0.49 \mu\text{m}$), green (0.49 to $0.58 \mu\text{m}$), yellow (0.58 to $0.60 \mu\text{m}$), orange (0.60 to $0.62 \mu\text{m}$), and red (0.62 to $0.78 \mu\text{m}$).

Solution: In the glass,

$$\lambda = \frac{\lambda_0}{\sqrt{\epsilon_r}} = \frac{0.61}{\sqrt{1.44}} = 0.508 \mu\text{m}.$$

The light would appear green.

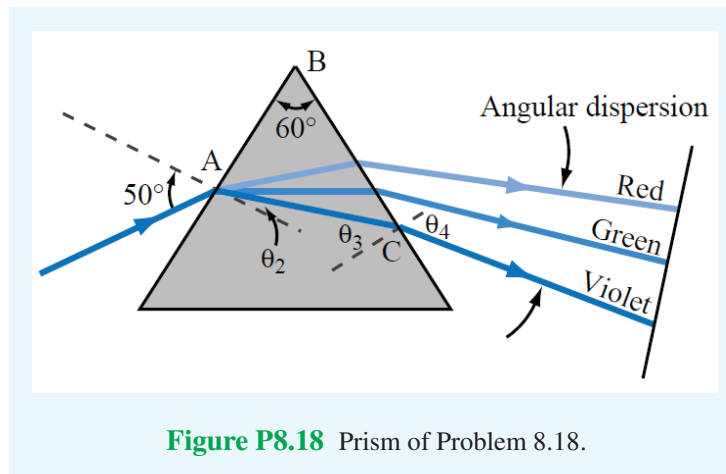
Q3. Sol'n.

8.18 For some types of glass, the index of refraction varies with wavelength. A prism made of a material with

$$n = 1.71 - \frac{4}{30} \lambda_0 \quad (\lambda_0 \text{ in } \mu\text{m}),$$

where λ_0 is the wavelength in vacuum, was used to disperse white light as shown in Fig. P8.18. The white light is incident at an angle of 50° , the wavelength λ_0 of red light is $0.7 \mu\text{m}$, and that of violet light is $0.4 \mu\text{m}$. Determine the angular dispersion in degrees.

Solution:



For violet,

$$n_v = 1.71 - \frac{4}{30} \times 0.4 = 1.66, \quad \sin \theta_2 = \frac{\sin \theta}{n_v} = \frac{\sin 50^\circ}{1.66},$$

or

$$\theta_2 = 27.48^\circ.$$

From the geometry of triangle ABC ,

$$180^\circ = 60^\circ + (90^\circ - \theta_2) + (90^\circ - \theta_3),$$

or

$$\theta_3 = 60^\circ - \theta_2 = 60 - 27.48^\circ = 32.52^\circ,$$

and

$$\sin \theta_4 = n_v \sin \theta_3 = 1.66 \sin 32.52^\circ = 0.89,$$

or

$$\theta_4 = 63.18^\circ.$$

For red,

$$n_r = 1.71 - \frac{4}{30} \times 0.7 = 1.62,$$

$$\theta_2 = \sin^{-1} \left[\frac{\sin 50^\circ}{1.62} \right] = 28.22^\circ,$$

$$\theta_3 = 60^\circ - 28.22^\circ = 31.78^\circ,$$

$$\theta_4 = \sin^{-1} [1.62 \sin 31.78^\circ] = 58.56^\circ.$$

Hence, angular dispersion $= 63.18^\circ - 58.56^\circ = 4.62^\circ$.

Q4 Sol'n.

8.19 A parallel-polarized plane wave is incident from air at an angle $\theta_i = 30^\circ$ onto a pair of dielectric layers as shown in Fig. P8.19.

- (a) Determine the angles of transmission θ_2 , θ_3 , and θ_4 .
- (b) Determine the lateral distance d .

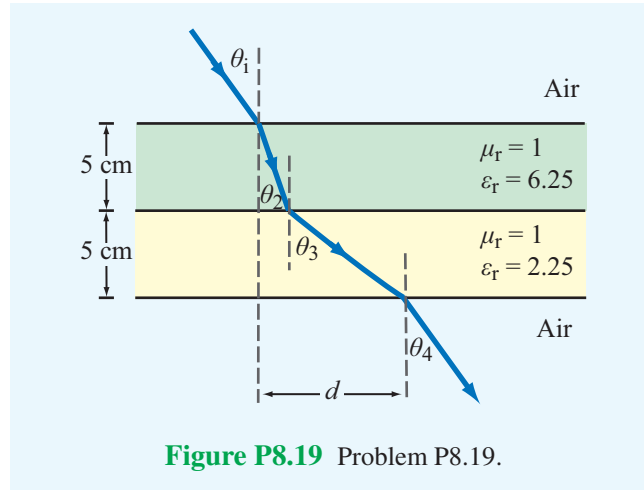


Figure P8.19 Problem P8.19.

Solution:

- (a) Application of Snell's law of refraction given by (8.31) leads to:

$$\sin \theta_2 = \sin \theta_i \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}} = \sin 30^\circ \sqrt{\frac{1}{6.25}} = 0.2$$
$$\theta_2 = 11.54^\circ.$$

Similarly,

$$\sin \theta_3 = \sin \theta_2 \sqrt{\frac{\epsilon_{r2}}{\epsilon_{r3}}} = \sin 11.54^\circ \sqrt{\frac{6.25}{2.25}} = 0.33$$
$$\theta_3 = 19.48^\circ.$$

And,

$$\sin \theta_4 = \sin \theta_3 \sqrt{\frac{\epsilon_{r3}}{\epsilon_{r4}}} = \sin 19.48^\circ \sqrt{\frac{2.25}{1}} = 0.5$$
$$\theta_4 = 30^\circ.$$

As expected, the exit ray back into air will be at the same angle as θ_i .

(b)

$$\begin{aligned}d &= (5 \text{ cm}) \tan \theta_2 + (5 \text{ cm}) \tan \theta_3 \\&= 5 \tan 11.54^\circ + 5 \tan 19.48^\circ = 2.79 \text{ cm}.\end{aligned}$$

Q5. Sol'n.

8.27 A plane wave in air with

$$\tilde{\mathbf{E}}_i = \hat{\mathbf{y}} 20e^{-j(3x+4z)} \quad (\text{V/m})$$

is incident upon the planar surface of a dielectric material, with $\epsilon_r = 4$, occupying the half-space $z \geq 0$. Determine:

- (a) The polarization of the incident wave.
- (b) The angle of incidence.
- (c) The time-domain expressions for the reflected electric and magnetic fields.
- (d) The time-domain expressions for the transmitted electric and magnetic fields.
- (e) The average power density carried by the wave in the dielectric medium.

Solution:

- (a) $\tilde{\mathbf{E}}_i = \hat{\mathbf{y}} 20e^{-j(3x+4z)} \text{ V/m}$.

Since $\mathbf{E}_i$ is along $\hat{\mathbf{y}}$, which is perpendicular to the plane of incidence, the wave is perpendicularly polarized.

- (b) From Eq. (8.48a), the argument of the exponential is

$$-jk_1(x \sin \theta_i + z \cos \theta_i) = -j(3x + 4z).$$

Hence,

$$k_1 \sin \theta_i = 3, \quad k_1 \cos \theta_i = 4,$$

from which we determine that

$$\tan \theta_i = \frac{3}{4} \quad \text{or} \quad \theta_i = 36.87^\circ,$$

and

$$k_1 = \sqrt{3^2 + 4^2} = 5 \quad (\text{rad/m}).$$

Also,

$$\omega = u_p k = ck = 3 \times 10^8 \times 5 = 1.5 \times 10^9 \quad (\text{rad/s}).$$

- (c)

$$\eta_1 = \eta_0 = 377 \, \Omega,$$

$$\eta_2 = \frac{\eta_0}{\sqrt{\epsilon_{r2}}} = \frac{\eta_0}{2} = 188.5 \, \Omega,$$

$$\theta_t = \sin^{-1} \left[\frac{\sin \theta_i}{\sqrt{\epsilon_{r2}}} \right] = \sin^{-1} \left[\frac{\sin 36.87^\circ}{\sqrt{4}} \right] = 17.46^\circ,$$

$$\Gamma_{\perp} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} = -0.41,$$

$$\tau_{\perp} = 1 + \Gamma_{\perp} = 0.59.$$

In accordance with Eq. (8.49a), and using the relation $E_0^r = \Gamma_{\perp} E_0^i$,

$$\begin{aligned}\tilde{\mathbf{E}}^r &= -\hat{\mathbf{y}} 8.2 e^{-j(3x-4z)}, \\ \tilde{\mathbf{H}}^r &= -(\hat{\mathbf{x}} \cos \theta_i + \hat{\mathbf{z}} \sin \theta_i) \frac{8.2}{\eta_0} e^{-j(3x-4z)},\end{aligned}$$

where we used the fact that $\theta_i = \theta_r$ and the z -direction has been reversed.

$$\begin{aligned}\mathbf{E}^r &= \Re[\tilde{\mathbf{E}}^r e^{j\omega t}] = -\hat{\mathbf{y}} 8.2 \cos(1.5 \times 10^9 t - 3x + 4z) \quad (\text{V/m}), \\ \mathbf{H}^r &= -(\hat{\mathbf{x}} 17.4 + \hat{\mathbf{z}} 13.06) \cos(1.5 \times 10^9 t - 3x + 4z) \quad (\text{mA/m}).\end{aligned}$$

(d) In medium 2,

$$k_2 = k_1 \sqrt{\frac{\epsilon_2}{\epsilon_1}} = 5\sqrt{4} = 10 \quad (\text{rad/m}),$$

and

$$\theta_t = \sin^{-1} \left[\sqrt{\frac{\epsilon_1}{\epsilon_2}} \sin \theta_i \right] = \sin^{-1} \left[\frac{1}{2} \sin 36.87^\circ \right] = 17.46^\circ$$

and the exponent of $\mathbf{E}^t$ and $\mathbf{H}^t$ is

$$-jk_2(x \sin \theta_t + z \cos \theta_t) = -j10(x \sin 17.46^\circ + z \cos 17.46^\circ) = -j(3x + 9.54z).$$

Hence,

$$\begin{aligned}\tilde{\mathbf{E}}^t &= \hat{\mathbf{y}} 20 \times 0.59 e^{-j(3x+9.54z)}, \\ \tilde{\mathbf{H}}^t &= (-\hat{\mathbf{x}} \cos \theta_t + \hat{\mathbf{z}} \sin \theta_t) \frac{20 \times 0.59}{\eta_2} e^{-j(3x+9.54z)}, \\ \mathbf{E}^t &= \Re[\tilde{\mathbf{E}}^t e^{j\omega t}] = \hat{\mathbf{y}} 11.8 \cos(1.5 \times 10^9 t - 3x - 9.54z) \quad (\text{V/m}), \\ \mathbf{H}^t &= (-\hat{\mathbf{x}} \cos 17.46^\circ + \hat{\mathbf{z}} \sin 17.46^\circ) \frac{11.8}{188.5} \cos(1.5 \times 10^9 t - 3x - 9.54z) \\ &= (-\hat{\mathbf{x}} 59.72 + \hat{\mathbf{z}} 18.78) \cos(1.5 \times 10^9 t - 3x - 9.54z) \quad (\text{mA/m}).\end{aligned}$$

(e)

$$S_{\text{av}}^t = \frac{|E_0^t|^2}{2\eta_2} = \frac{(11.8)^2}{2 \times 188.5} = 0.36 \quad (\text{W/m}^2).$$

Q6. Sol'n.

8.28 Repeat Problem 8.27 for a wave in air with

$$\tilde{\mathbf{H}}^i = \hat{\mathbf{y}} 2 \times 10^{-2} e^{-j(8x+6z)} \quad (\text{A/m})$$

incident upon the planar boundary of a dielectric medium ($z \geq 0$) with $\epsilon_r = 9$.

Solution:

(a) $\tilde{\mathbf{H}}^i = \hat{\mathbf{y}} 2 \times 10^{-2} e^{-j(8x+6z)}.$

Since $\mathbf{H}^i$ is along $\hat{\mathbf{y}}$, which is perpendicular to the plane of incidence, the wave is TM polarized, or equivalently, its electric field vector is parallel polarized (parallel to the plane of incidence).

(b) From Eq. (8.65b), the argument of the exponential is

$$-jk_1(x \sin \theta_i + z \cos \theta_i) = -j(8x + 6z).$$

Hence,

$$k_1 \sin \theta_i = 8, \quad k_1 \cos \theta_i = 6,$$

from which we determine

$$\theta_i = \tan^{-1} \left(\frac{8}{6} \right) = 53.13^\circ,$$

$$k_1 = \sqrt{6^2 + 8^2} = 10 \quad (\text{rad/m}).$$

Also,

$$\omega = u_p k = ck = 3 \times 10^8 \times 10 = 3 \times 10^9 \quad (\text{rad/s}).$$

(c)

$$\eta_1 = \eta_0 = 377 \, \Omega,$$

$$\eta_2 = \frac{\eta_0}{\sqrt{\epsilon_{r2}}} = \frac{\eta_0}{3} = 125.67 \, \Omega,$$

$$\theta_t = \sin^{-1} \left[\frac{\sin \theta_i}{\sqrt{\epsilon_{r2}}} \right] = \sin^{-1} \left[\frac{\sin 53.13^\circ}{\sqrt{9}} \right] = 15.47^\circ,$$

$$\Gamma_{\parallel} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} = -0.30,$$

$$\tau_{\parallel} = (1 + \Gamma_{\parallel}) \frac{\cos \theta_i}{\cos \theta_t} = 0.44.$$

In accordance with Eqs. (8.65a) to (8.65d), $E_0^i = 2 \times 10^{-2} \eta_1$ and

$$\tilde{\mathbf{E}}^i = (\hat{\mathbf{x}} \cos \theta_i - \hat{\mathbf{z}} \sin \theta_i) 2 \times 10^{-2} \eta_1 e^{-j(8x+6z)} = (\hat{\mathbf{x}} 4.52 - \hat{\mathbf{z}} 6.03) e^{-j(8x+6z)}.$$

$\tilde{\mathbf{E}}^r$ is similar to $\tilde{\mathbf{E}}^i$ except for reversal of z -components and multiplication of amplitude by $\Gamma_{\parallel}$. Hence, with $\Gamma_{\parallel} = -0.30$,

$$\begin{aligned}\mathbf{E}^r &= \Re\{\tilde{\mathbf{E}}^r e^{j\omega t}\} = -(\hat{\mathbf{x}} 1.36 + \hat{\mathbf{z}} 1.81) \cos(3 \times 10^9 t - 8x + 6z) \text{ V/m}, \\ \mathbf{H}^r &= \hat{\mathbf{y}} 2 \times 10^{-2} \Gamma_{\parallel} \cos(3 \times 10^9 t - 8x + 6z) \\ &= -\hat{\mathbf{y}} 0.6 \times 10^{-2} \cos(3 \times 10^9 t - 8x + 6z) \text{ A/m}.\end{aligned}$$

(d) In medium 2,

$$\begin{aligned}k_2 &= k_1 \sqrt{\frac{\epsilon_2}{\epsilon_1}} = 10\sqrt{9} = 30 \text{ rad/m}, \\ \theta_t &= \sin^{-1} \left[\sqrt{\frac{\epsilon_2}{\epsilon_1}} \sin \theta_i \right] = \sin^{-1} \left[\frac{1}{3} \sin 53.13^\circ \right] = 15.47^\circ,\end{aligned}$$

and the exponent of $\mathbf{E}^t$ and $\mathbf{H}^t$ is

$$-jk_2(x \sin \theta_t + z \cos \theta_t) = -j30(x \sin 15.47^\circ + z \cos 15.47^\circ) = -j(8x + 28.91z).$$

Hence,

$$\begin{aligned}\tilde{\mathbf{E}}^t &= (\hat{\mathbf{x}} \cos \theta_t - \hat{\mathbf{z}} \sin \theta_t) E_0^i \tau_{\parallel} e^{-j(8x+28.91z)} \\ &= (\hat{\mathbf{x}} 0.96 - \hat{\mathbf{z}} 0.27) 2 \times 10^{-2} \times 377 \times 0.44 e^{-j(8x+28.91z)} \\ &= (\hat{\mathbf{x}} 3.18 - \hat{\mathbf{z}} 0.90) e^{-j(8x+28.91z)}, \\ \tilde{\mathbf{H}}^t &= \hat{\mathbf{y}} \frac{E_0^i \tau_{\parallel}}{\eta_2} e^{-j(8x+28.91z)} \\ &= \hat{\mathbf{y}} 2.64 \times 10^{-2} e^{-j(8x+28.91z)}, \\ \mathbf{E}^t &= \Re\{\tilde{\mathbf{E}}^t e^{j\omega t}\} \\ &= (\hat{\mathbf{x}} 3.18 - \hat{\mathbf{z}} 0.90) \cos(3 \times 10^9 t - 8x - 28.91z) \text{ V/m}, \\ \mathbf{H}^t &= \hat{\mathbf{y}} 2.64 \times 10^{-2} \cos(3 \times 10^9 t - 8x - 28.91z) \text{ A/m}.\end{aligned}$$

(e)

$$S_{\text{av}}^t = \frac{|E_0^t|^2}{2\eta_2} = \frac{|H_0^t|^2}{2} \eta_2 = \frac{(2.64 \times 10^{-2})^2}{2} \times 125.67 = 44 \text{ mW/m}^2.$$

Q7. Sol'n.

8.31 Natural light is randomly polarized, which means that, on average, half the light energy is polarized along any given direction (in the plane orthogonal to the direction of propagation) and the other half of the energy is polarized along the direction orthogonal to the first polarization direction. Hence, when treating natural light incident upon a planar boundary, we can consider half of its energy to be in the form of parallel-polarized waves and the other half as perpendicularly polarized waves. Determine the fraction of the incident power reflected by the planar surface of a piece of glass with $n = 1.5$ when illuminated by natural light at 70° .

Solution: Assume the incident power is 1 W. Hence:

Incident power with parallel polarization = 0.5 W,

Incident power with perpendicular polarization = 0.5 W.

$\epsilon_2/\epsilon_1 = (n_2/n_1)^2 = n^2 = 1.5^2 = 2.25$. Equations (8.60) and (8.68) give

$$\Gamma_{\perp} = \frac{\cos 70^\circ - \sqrt{2.25 - \sin^2 70^\circ}}{\cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}} = -0.55,$$
$$\Gamma_{\parallel} = \frac{-2.25 \cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}}{2.25 \cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}} = 0.21.$$

Reflected power with parallel polarization = $0.5 (\Gamma_{\parallel})^2$
= $0.5 (0.21)^2 = 22 \text{ mW}$,

Reflected power with perpendicular polarization = $0.5 (\Gamma_{\perp})^2$
= $0.5 (0.55)^2 = 151.3 \text{ mW}$.

Total reflected power = $22 + 151.3 = 173.3 \text{ mW}$, or 17.33%..

Q8. Sol'n.

8.35 A parallel-polarized beam of light with an electric field amplitude of 10 (V/m) is incident in air on polystyrene with $\mu_r = 1$ and $\epsilon_r = 2.6$. If the incidence angle at the air-polystyrene planar boundary is 60° , determine the following:

- (a) The reflectivity and transmissivity.
- (b) The power carried by the incident, reflected, and transmitted beams if the spot on the boundary illuminated by the incident beam is 1 m^2 in area.

Solution:

- (a) From Eq. (8.68),

$$\begin{aligned}\Gamma_{\parallel} &= \frac{-(\epsilon_2/\epsilon_1) \cos \theta_i + \sqrt{(\epsilon_2/\epsilon_1) - \sin^2 \theta_i}}{(\epsilon_2/\epsilon_1) \cos \theta_i + \sqrt{(\epsilon_2/\epsilon_1) - \sin^2 \theta_i}} \\ &= \frac{-2.6 \cos 60^\circ + \sqrt{2.6 - \sin^2 60^\circ}}{2.6 \cos 60^\circ + \sqrt{2.6 - \sin^2 60^\circ}} = -0.0226, \\ R_{\parallel} &= |\Gamma_{\parallel}|^2 = (0.0226)^2 = 5.1 \times 10^{-4}, \\ T_{\parallel} &= 1 - R_{\parallel} = 0.99949.\end{aligned}$$

- (b)

$$\begin{aligned}P_{\parallel}^i &= \frac{|E_{\parallel 0}^i|^2}{2\eta_1} A \cos \theta_i = \frac{(10)^2}{2 \times 120\pi} \times \cos 60^\circ = 66 \text{ mW}, \\ P_{\parallel}^r &= R_{\parallel} P_{\parallel}^i = (5.1 \times 10^{-4}) \times 0.066 = 0.033 \text{ mW}, \\ P_{\parallel}^t &= T_{\parallel} P_{\parallel}^i = 0.99949 \times 0.066 = 65.97 \text{ mW}.\end{aligned}$$

Q9. Sol'n.

8.36 A 50-MHz right-hand circularly polarized plane wave with an electric field modulus of 30 V/m is normally incident in air upon a dielectric medium with $\epsilon_r = 9$ and occupying the region defined by $z \geq 0$.

- (a) Write an expression for the electric field phasor of the incident wave, given that the field is a positive maximum at $z = 0$ and $t = 0$.
- (b) Calculate the reflection and transmission coefficients.
- (c) Write expressions for the electric field phasors of the reflected wave, the transmitted wave, and the total field in the region $z \leq 0$.
- (d) Determine the percentages of the incident average power reflected by the boundary and transmitted into the second medium.

Solution:

(a)

$$k_1 = \frac{\omega}{c} = \frac{2\pi \times 50 \times 10^6}{3 \times 10^8} = \frac{\pi}{3} \text{ rad/m},$$

$$k_2 = \frac{\omega}{c} \sqrt{\epsilon_{r2}} = \frac{\pi}{3} \sqrt{9} = \pi \text{ rad/m}.$$

From (7.57), RHC wave traveling in $+z$ direction:

$$\begin{aligned} \tilde{\mathbf{E}}^i &= a_0(\hat{\mathbf{x}} + \hat{\mathbf{y}} e^{-j\pi/2}) e^{-jk_1 z} = a_0(\hat{\mathbf{x}} - j\hat{\mathbf{y}}) e^{-jk_1 z} \\ \mathbf{E}^i(z, t) &= \Re \left[\tilde{\mathbf{E}}^i e^{j\omega t} \right] \\ &= \Re \left[a_0(\hat{\mathbf{x}} e^{j(\omega t - k_1 z)} + \hat{\mathbf{y}} e^{j(\omega t - k_1 z - \pi/2)}) \right] \\ &= \hat{\mathbf{x}} a_0 \cos(\omega t - k_1 z) + \hat{\mathbf{y}} a_0 \cos(\omega t - k_1 z - \pi/2) \\ &= \hat{\mathbf{x}} a_0 \cos(\omega t - k_1 z) + \hat{\mathbf{y}} a_0 \sin(\omega t - k_1 z) \\ |\mathbf{E}^i| &= [a_0^2 \cos^2(\omega t - k_1 z) + a_0^2 \sin^2(\omega t - k_1 z)]^{1/2} = a_0 = 30 \text{ V/m}. \end{aligned}$$

Hence,

$$\tilde{\mathbf{E}}^i = 30(\hat{\mathbf{x}} - j\hat{\mathbf{y}}) e^{-j\pi z/3} \quad (\text{V/m}).$$

(b)

$$\begin{aligned} \eta_1 &= \eta_0 = 120\pi \quad (\Omega), \quad \eta_2 = \frac{\eta_0}{\sqrt{\epsilon_{r2}}} = \frac{120\pi}{\sqrt{9}} = 40\pi \quad (\Omega). \\ \Gamma &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{40\pi - 120\pi}{40\pi + 120\pi} = -0.5 \\ \tau &= 1 + \Gamma = 1 - 0.5 = 0.5. \end{aligned}$$

(c)

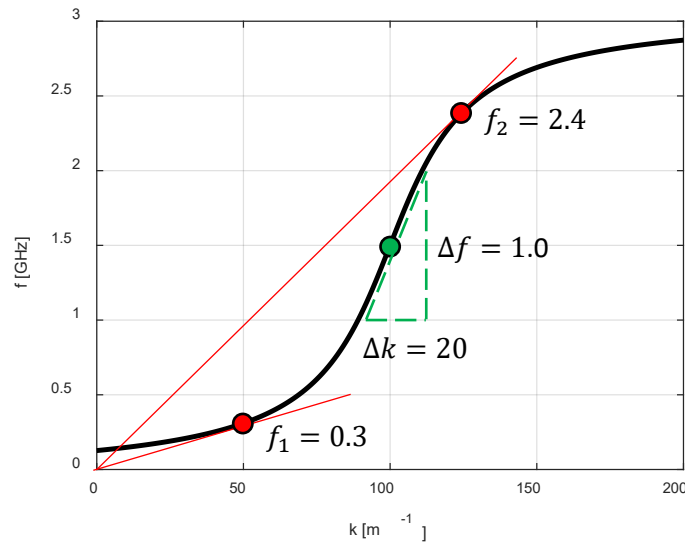
$$\begin{aligned}\tilde{\mathbf{E}}^r &= \Gamma a_0 (\hat{\mathbf{x}} - j\hat{\mathbf{y}}) e^{jk_1 z} \\ &= -0.5 \times 30 (\hat{\mathbf{x}} - j\hat{\mathbf{y}}) e^{jk_1 z} \\ &= -15 (\hat{\mathbf{x}} - j\hat{\mathbf{y}}) e^{j\pi z/3} \quad (\text{V/m}). \\ \tilde{\mathbf{E}}^t &= \tau a_0 (\hat{\mathbf{x}} - j\hat{\mathbf{y}}) e^{-jk_2 z} \\ &= 15 (\hat{\mathbf{x}} - j\hat{\mathbf{y}}) e^{-j\pi z} \quad (\text{V/m}). \\ \tilde{\mathbf{E}}_1 &= \tilde{\mathbf{E}}^i + \tilde{\mathbf{E}}^r \\ &= 30 (\hat{\mathbf{x}} - j\hat{\mathbf{y}}) e^{-j\pi z/3} - 15 (\hat{\mathbf{x}} - j\hat{\mathbf{y}}) e^{j\pi z/3} \\ &= 15 (\hat{\mathbf{x}} - j\hat{\mathbf{y}}) [2e^{-j\pi z/3} - e^{j\pi z/3}] \quad (\text{V/m}).\end{aligned}$$

(d)

$$\% \text{ of reflected power} = 100 \times |\Gamma|^2 = 100 \times (0.5)^2 = 25\%$$

$$\% \text{ of transmitted power} = 100 |\tau|^2 \frac{\eta_1}{\eta_2} = 100 \times (0.5)^2 \times \frac{120\pi}{40\pi} = 75\%.$$

10. (Adapted from the MIT OCW Lecture on EM Waves) The figure below shows the dispersion curve for a certain medium. (Do not make any assumptions about the mathematical form of this curve.)



- For approximately what frequency(ies) are the phase and group velocities equal?
- For which frequency range will the group velocity be faster than the phase velocity?
- What frequency would you choose for transmitting a pulse at the highest possible speed? How big is this speed?

Solution:

Note that $v_p = \frac{2\pi f}{k} = 2\pi \times \text{slope of a line connecting the point of the curve to the origin}$;

$v_g = \left(\frac{dk}{d\omega}\right)^{-1} = \left(\frac{1}{2\pi} \frac{dk}{df}\right)^{-1} = 2\pi \times \text{slope of the tangent line to the point on the curve}$.

- For the two points $f_1 = 0.3$ GHz and $f_2 = 2.4$ GHz (labelled in red), the tangent lines also reach the origin, so $v_p = v_g$.
- For the frequency range $f_1 < f < f_2$, the tangent has a higher slope than the line connecting the curve to the origin, so for this frequency range $v_g > v_p$.
- The curve tangent has the largest slope at $f = 1.5$ GHz (labelled in green), so the group velocity (ie. the pulse speed) is the fastest for this frequency. The velocity is

$$v_g = \left(\frac{dk}{d\omega}\right)^{-1} = 2\pi \frac{\Delta f}{\Delta k} \Big|_{\text{tangent}} = \frac{2\pi(1.0 \times 10^9)}{20} = 3.2 \times 10^8 \text{ [m/s]}.$$