

EE2108

Lecture 5:

Part-1: System of Linear Equations

Part-2: Eigenvalues and Eigenvectors

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Solving Systems of Linear Equations

What if matrix A is **not** diagonal, upper triangular or lower triangular?

We can find the solutions via the direct or indirect methods.

1. Direct methods: They transform the problem to an easy-to-solve problem!
 - Gaussian elimination
 - LU factorization
2. Indirect methods
 - Gauss-Seidel iteration

LU Decomposition Approach

It was introduced by Polish mathematician Tadeusz Banachiewicz in 1938.



Tadeusz Banachiewicz

LU Decomposition

- Suppose that matrix **A** can be written as

$$A = LU$$

- L is lower triangular
- U is upper triangular.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

All the diagonal entries of L are 1.

- This is called the **LU decomposition** of A .

$$\underbrace{\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}}_{\text{Matrix } A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\text{Vector } \mathbf{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{\text{Vector } \mathbf{b}}$$

$\mathbf{Ax} = \mathbf{b}$

LU Decomposition Approach

$$A\mathbf{x} = \mathbf{b}$$

- Let's write our equation as

$$L\mathbf{U}\mathbf{x} = \mathbf{b}$$

- Let's use $\mathbf{Y} = \mathbf{U}\mathbf{x}$ to rewrite our equation one more time.
$$\underbrace{L\mathbf{U}\mathbf{x}}_{\mathbf{Y} = \mathbf{U}\mathbf{x}} = \mathbf{b} \longrightarrow L\mathbf{Y} = \mathbf{b}$$

- We get a new equation with $\mathbf{Y}$ as **unknown**. We can find $\mathbf{Y}$ by **forward substitution**. **Why?**
$$L\mathbf{Y} = \mathbf{b}$$
- Remember, $\mathbf{U}\mathbf{x} = \mathbf{Y}$. We can find $\mathbf{x}$ by **backward substitution**. **Why?**
$$\mathbf{U}\mathbf{x} = \mathbf{Y}$$

LU Decomposition Approach

Step 1: Find the LU decomposition of A. That is, $A=LU$.

Step 2: Find Y by solving $LY=b$ using forward substitution.

Step 3: Find x by solving $Ux=Y$ using backward substitution.

LU Decomposition

How to find the LU decomposition of A? (Step 1)

Assume, $A=LU$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$$

Lower triangular.
Diagonal entries equal to 1.

$$\mathbf{U} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Upper triangular

LU Decomposition

- $A=LU$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Lower triangular.

Diagonal entries equal to 1.

Upper triangular

- If we multiply L by U, we have

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{11}L_{21} & U_{12}L_{21} + U_{22} & U_{13}L_{21} + U_{23} \\ U_{11}L_{31} & U_{12}L_{31} + U_{22}L_{32} & U_{13}L_{31} + U_{23}L_{32} + U_{33} \end{bmatrix}$$

LU Decomposition

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

- Remember, matrix A is known.
- We can find L and U from the following equation.

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{11}L_{21} & U_{12}L_{21} + U_{22} & U_{13}L_{21} + U_{23} \\ U_{11}L_{31} & U_{12}L_{31} + U_{22}L_{32} & U_{13}L_{31} + U_{23}L_{32} + U_{33} \end{bmatrix}$$

1- $U_{11} = A_{11}$	4- $L_{21} = \frac{A_{21}}{U_{11}}$	7- $L_{31} = \frac{A_{31}}{U_{11}}$
2- $U_{12} = A_{12}$	5- $U_{22} = A_{22} - U_{12}L_{21}$	8- $L_{32} = \frac{A_{32} - U_{12}L_{31}}{U_{22}}$
3- $U_{13} = A_{13}$	6- $U_{23} = A_{23} - U_{13}L_{21}$	9- $U_{33} = A_{33} - U_{13}L_{31} - U_{23}L_{32}$

Example

Solve the system of equations $Ax=b$ using LU decomposition.

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 6 & -1 \\ 2 & -1 & 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 13 \\ 5 \end{bmatrix}$$

Example

First, we find the LU decomposition of A.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 6 & -1 \\ 2 & -1 & 2 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A_{11} = 1, A_{12} = 4, A_{13} = 1, A_{21} = 1, A_{22} = 6, A_{23} = -1, A_{31} = 2, A_{32} = -1, A_{33} = 2$$

$$\begin{aligned} U_{11} &= A_{11} = 1 & U_{22} &= A_{22} - U_{12}L_{21} = 6 - 4 \times 1 = 2 & U_{33} &= A_{33} - U_{13}L_{31} - U_{23}L_{32} \\ & & & & &= 2 - 1 \times 2 - (-2)(-4.5) = -9 \\ U_{12} &= A_{12} = 4 & U_{23} &= A_{23} - U_{13}L_{21} = -1 - 1 \times 1 = -2 \\ U_{13} &= A_{13} = 1 & L_{31} &= \frac{A_{31}}{U_{11}} = 2 \\ L_{21} &= \frac{A_{21}}{U_{11}} = \frac{1}{1} & L_{32} &= \frac{A_{32} - U_{12}L_{31}}{U_{22}} = \frac{-1 - 4 \times 2}{2} = -4.5 \end{aligned}$$

Example

First, we find the LU decomposition of A.

$$U_{11} = 1$$

$$U_{22} = 2$$

$$L_{32} = -4.5$$

$$U_{12} = 4$$

$$U_{23} = -2$$

$$L_{21} = 1$$

$$U_{13} = 1$$

$$U_{33} = -9$$

$$L_{31} = 2$$

$$\mathbf{U} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & -9 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -4.5 & 1 \end{bmatrix}$$

Example cont.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 6 & -1 \\ 2 & -1 & 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 13 \\ 5 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -4.5 & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & -9 \end{bmatrix}$$

Then, we solve $\mathbf{LY}=\mathbf{b}$ using forward substitution.

$$[\mathbf{L} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 1 & 1 & 0 & 13 \\ 2 & -4.5 & 1 & 5 \end{array} \right] \quad \leftarrow \text{Form the augmented matrix using L and b}$$

Forward substitution

$$y_1 = 7$$

$$y_1 = 7$$

$$y_1 + y_2 = 13$$

$$y_2 = 13 - y_1 = 13 - 7 = 6$$

$$2y_1 - 4.5y_2 + y_3 = 5$$

$$y_3 = 5 - 2y_1 + 4.5y_2 = 5 - 2(7) + 4.5(6) = 18$$

Example cont.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 6 & -1 \\ 2 & -1 & 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 13 \\ 5 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -4.5 & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & -9 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 7 \\ 6 \\ 8 \end{bmatrix}$$

Finally, we solve $\mathbf{Ux}=\mathbf{y}$ using backward substitution.

$$[\mathbf{U} \mid \mathbf{y}] = \left[\begin{array}{ccc|c} 1 & 4 & 1 & 7 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & -9 & 18 \end{array} \right] \quad \leftarrow \text{Form the augmented matrix using U and y}$$

Backward substitution

$$-9x_3 = 18$$

$$x_3 = \frac{18}{-9} = -2$$

$$2x_2 - 2x_3 = 6$$

$$x_2 = \frac{6 + 2x_3}{2} = \frac{6 + 2(-2)}{2} = 1$$

$$x_1 + 4x_2 + x_3 = 7$$

$$x_1 = 7 - 4x_2 - x_3 = 7 - 4(1) - (-2) = 5$$

LU Decomposition

- $A=LU$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Lower triangular.

Diagonal entries equal to 1.

Upper triangular

- If we multiply L by U, we have

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{11}L_{21} & U_{12}L_{21} + U_{22} & U_{13}L_{21} + U_{23} \\ U_{11}L_{31} & U_{12}L_{31} + U_{22}L_{32} & U_{13}L_{31} + U_{23}L_{32} + U_{33} \end{bmatrix}$$

LU Decomposition

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{11}L_{21} & U_{12}L_{21} + U_{22} & U_{13}L_{21} + U_{23} \\ U_{11}L_{31} & U_{12}L_{31} + U_{22}L_{32} & U_{13}L_{31} + U_{23}L_{32} + U_{33} \end{bmatrix}$$

$$U_{11} = A_{11}$$

$$U_{12} = A_{12}$$

$$U_{13} = A_{13}$$

$$U_{23} + U_{13}L_{21} = A_{23} \Rightarrow U_{23} = A_{23} - U_{13}L_{21}$$

$$U_{11}L_{31} = A_{31} \Rightarrow L_{31} = \frac{A_{31}}{U_{11}}$$

$$U_{22}L_{32} + U_{12}L_{31} = A_{32} \Rightarrow L_{32} = \frac{A_{32} - U_{12}L_{31}}{U_{22}}$$

$$U_{33} + U_{13}L_{31} + U_{23}L_{32} = A_{33} \Rightarrow U_{33} = A_{33} - U_{13}L_{31} - U_{23}L_{32}$$

$$U_{11}L_{21} = A_{21} \Rightarrow L_{21} = \frac{A_{21}}{U_{11}}$$

$$U_{22} + U_{12}L_{21} = A_{22} \Rightarrow U_{22} = A_{22} - U_{12}L_{21}$$

LU Decomposition

$$1- U_{11} = A_{11}$$

$$2- U_{12} = A_{12}$$

$$3- U_{13} = A_{13}$$

$$4- L_{21} = \frac{A_{21}}{U_{11}}$$

$$5- U_{22} = A_{22} - U_{12}L_{21}$$

$$6- U_{23} = A_{23} - U_{13}L_{21}$$

$$7- L_{31} = \frac{A_{31}}{U_{11}}$$

$$8- L_{32} = \frac{A_{32} - U_{12}L_{31}}{U_{22}}$$

$$9- U_{33} = A_{33} - U_{13}L_{31} - U_{23}L_{32}$$

Solving Systems of Linear Equations

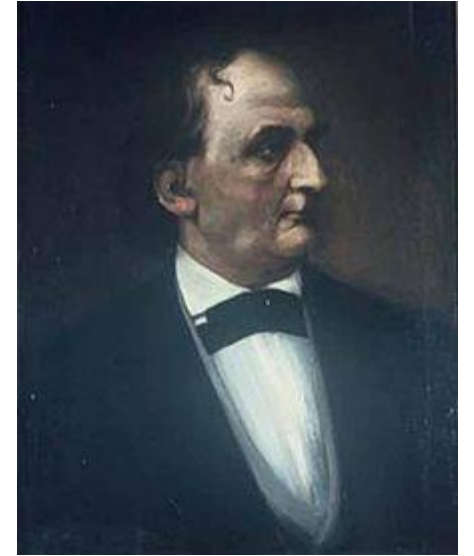
What if matrix A is **not** diagonal, upper triangular or lower triangular?

We can find the solutions via the direct or indirect methods.

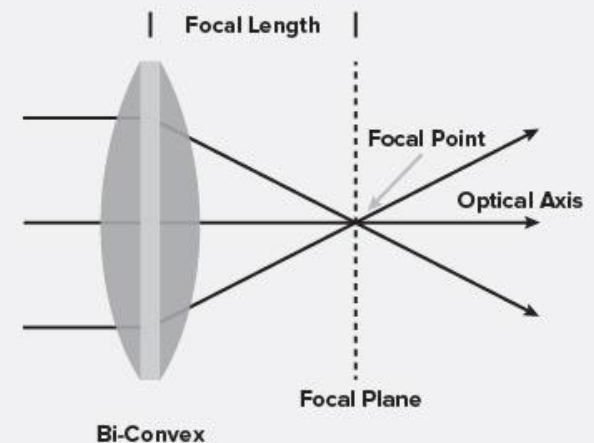
1. Direct methods: They transform the problem to an easy-to-solve problem!
 - Gaussian elimination
 - LU factorization
2. Indirect methods
 - Gauss-Seidel iteration

Gauss-Seidel Method

- A German mathematician.
- Seidel worked on different areas of mathematics and physics.
- He applied **probability** to **astronomy**.
- He identified **parameters** describing the **aberration** of a lens.
- They are now called **Seidel sums**.



Philipp Ludwig von Seidel



Gauss-Seidel Method

1. Find the matrix $\tilde{A}$ and vector $\tilde{b}$.
2. Select an arbitrary vector X_0 . (n-dimensional)
3. $k = 1$.
4. Compute X_k according to $X_k = \tilde{b} - \tilde{A}X_{k-1}$.
5. Compute d_k according to $d_k = |X_k - X_{k-1}|$.
6. If all the elements of d_k are less than ϵ , stop.
7. Otherwise, $k \leftarrow k + 1$ and go to 4.

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

The last (generated) X_k is the solution.

Gauss-Seidel Method

Divide the first row by A_{11} $\rightarrow$

Divide the second row by A_{22} $\rightarrow$

$A =$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \dots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \dots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \dots & A_{nn} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{A_{12}}{A_{11}} & \frac{A_{13}}{A_{11}} & \dots & \frac{A_{1n}}{A_{11}} \\ \frac{A_{21}}{A_{22}} & 1 & \frac{A_{23}}{A_{22}} & \dots & \frac{A_{2n}}{A_{22}} \\ \frac{A_{31}}{A_{33}} & \frac{A_{32}}{A_{33}} & 1 & \dots & \frac{A_{3n}}{A_{33}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{A_{n1}}{A_{nn}} & \frac{A_{n2}}{A_{nn}} & \frac{A_{n3}}{A_{nn}} & \dots & 1 \end{bmatrix}$$

Divide the last row by A_{nn} $\rightarrow$

Matrix $\tilde{A} =$

$$\begin{bmatrix} 0 & \frac{A_{12}}{A_{11}} & \frac{A_{13}}{A_{11}} & \dots & \frac{A_{1n}}{A_{11}} \\ \frac{A_{21}}{A_{22}} & 0 & \frac{A_{23}}{A_{22}} & \dots & \frac{A_{2n}}{A_{22}} \\ \frac{A_{31}}{A_{33}} & \frac{A_{32}}{A_{33}} & 0 & \dots & \frac{A_{3n}}{A_{33}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{A_{n1}}{A_{nn}} & \frac{A_{n2}}{A_{nn}} & \frac{A_{n3}}{A_{nn}} & \dots & 0 \end{bmatrix}$$

Set the diagonal elements to zero.

Gauss-Seidel Method

Compute the vector $\tilde{b}$

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Divide by A_{11} $\longrightarrow$

Divide by A_{22} $\longrightarrow$

Divide by A_{nn} $\longrightarrow$

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \longrightarrow \tilde{b} = \begin{bmatrix} \frac{b_1}{A_{11}} \\ \frac{b_2}{A_{22}} \\ \vdots \\ \frac{b_n}{A_{nn}} \end{bmatrix}$$

Gauss-Seidel Method

1. Find the matrix $\tilde{A}$ and vector $\tilde{b}$.
2. Select an arbitrary vector X_0 . (n-dimensional)
3. $k = 1$.
4. Compute X_k according to $X_k = \tilde{b} - \tilde{A}X_{k-1}$.
5. Compute d_k according to $d_k = |X_k - X_{k-1}|$.
6. If all the elements of d_k are less than ϵ , stop.
7. Otherwise, $k \leftarrow k + 1$ and go to 4.

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

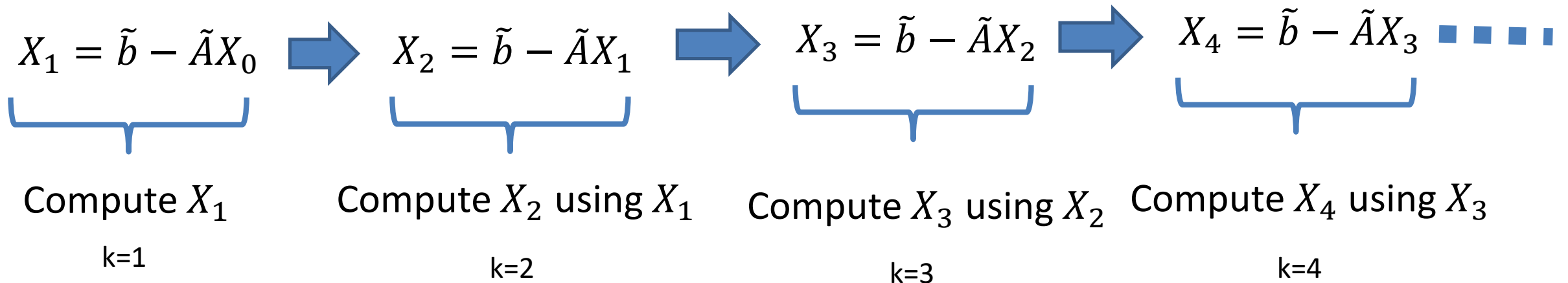
The last (generated) X_k is the solution.

Gauss-Seidel Method cont.

- Gauss-Seidel method is an **iterative way** to solve linear equations.
- **Iterative**: a set of computations are **repeated** over and over!

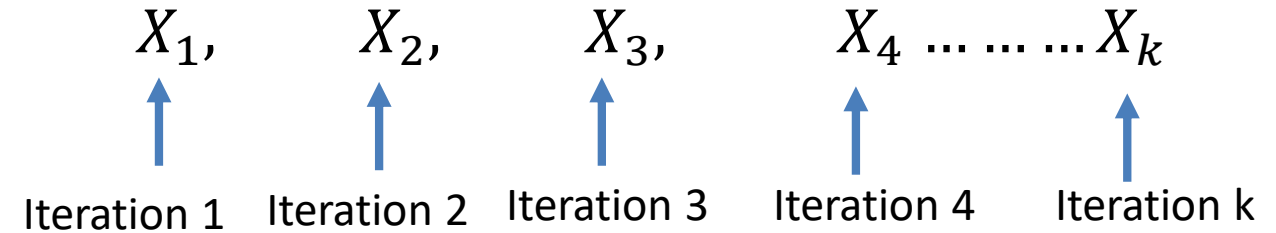
$$X_k = \tilde{b} - \tilde{A}X_{k-1}$$

- Start with an **arbitrary** vector $X_0 = [x_1^0, \dots, x_n^0]$.



Gauss-Seidel Method cont.

- Each iteration generates a new vector.



- Compute the **absolute difference** between **consecutive vectors**.

$$d_1 = |X_0 - X_1|, d_2 = |X_1 - X_2|, d_3 = |X_3 - X_2|, \dots$$

Difference vectors

- Stop at the **first time** that **all elements** of the **difference vector** are less than ϵ .
- The lastly generated X_k is the solution.

Example

Solve the following system of linear equations using the Gauss-Seidel method with $\epsilon = 10^{-3}$.

$$Ax = b, A = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -1 \\ 1 & -1 & 3 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -1 \\ 1 & -1 & 3 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Step 1: Form the matrix $\tilde{A}$

$$\begin{bmatrix} 1 & -0.5 & 0.25 \\ -0.5 & 1 & -0.25 \\ 1 & 1 & 1 \\ \frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix} \rightarrow \tilde{A} = \begin{bmatrix} 0 & -0.5 & 0.25 \\ -0.5 & 0 & -0.25 \\ 1 & 1 & 1 \\ \frac{1}{3} & -\frac{1}{3} & 0 \end{bmatrix}$$

Step 2: Form the vector $\tilde{b}$

$$\tilde{b} = \begin{bmatrix} 0.25 \\ 0.5 \\ 1 \\ 1 \end{bmatrix}$$

Example

- Initial vector: $X_0 = [0,0,0]$.
- The vector generated in iteration k: $X_k = [x_k^1, x_k^2, x_k^3]$.

$$X_k = \tilde{b} - \tilde{A}X_{k-1}$$

Gauss-Seidel iterations

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
k (Iteration)	1	2	3	4	5	6	7	8	9	10
x_k^1	0.2500	0.2500	0.4167	0.3958	0.4653	0.4427	0.4734	0.4582	0.4728	0.4640
x_k^2	0.5000	0.8750	0.8958	1.0104	0.9878	1.0339	1.0149	1.0360	1.0242	1.0346
x_k^3	1	1.0833	1.2083	1.1597	1.2049	1.1742	1.1970	1.1805	1.1926	1.1838

$$X_1 = \tilde{b} - \tilde{A}X_0 \Rightarrow X_2 = \tilde{b} - \tilde{A}X_1 \Rightarrow X_3 = \tilde{b} - \tilde{A}X_2 \Rightarrow X_4 = \tilde{b} - \tilde{A}X_3 \dots$$

Example

- The algorithm **converges** when the element of X_k do not **change** much **between** iterations.

	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}
	↓	↓	↓	↓	↓	↓	↓	↓
k (Iteration)	11	12	13	14	15	16	17	18
x_k^1	0.4713	0.4664	0.4702	0.4676	0.4696	0.4681	0.4692	0.4684
x_k^2	1.0279	1.0332	1.0296	1.0323	1.0304	1.0318	1.0308	1.0316
x_k^3	1.1902	1.1855	1.1889	1.1865	1.1883	1.1869	1.1879	1.1872

$$X_1 = \tilde{b} - \tilde{A}X_0 \Rightarrow X_2 = \tilde{b} - \tilde{A}X_1 \Rightarrow X_3 = \tilde{b} - \tilde{A}X_2 \Rightarrow X_4 = \tilde{b} - \tilde{A}X_3 \dots$$

Example

- The difference vector in iteration k : $d_k = [d_k^1, d_k^2, d_k^3]$.

$$d_k = |X_k - X_{k-1}|$$

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}
k (Iteration)	1	2	3	4	5	6	7	8	9	10
d_k^1	0.2500	0	0.1667	0.0208	0.0694	0.0226	0.0307	0.0152	0.0147	0.0089
d_k^2	0.5000	0.3750	0.0208	0.1146	0.0226	0.0460	0.0190	0.0211	0.0117	0.0104
d_k^3	1	0.0833	0.1250	0.0486	0.0451	0.0307	0.0229	0.0165	0.0121	0.0088

	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}
k (Iteration)	11	12	13	14	15	16	17	18
d_k^1	0.0074	0.0049	0.0038	0.0027	0.0020	0.0014	0.0011	7.6188e-04
d_k^2	0.0066	0.0053	0.0036	0.0028	0.0020	0.0015	0.0010	7.6760e-04
d_k^3	0.0064	0.0047	0.0034	0.0025	0.0018	0.0013	9.6008e-04	6.9963e-04

X_{18}



18
0.4684
1.0316
1.1872

At iteration 18, all the components of the difference vector are below $\epsilon = 10^{-3}$!

Solution

Gauss-Seidel Method

When does it work?

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

- When matrix A is diagonally dominant (**sufficient** condition).
- That is, **in each row**, the **absolute value** of the **diagonal element** is **more than** the **sum of absolute values** of **other elements** in **that row**.

$$|A_{11}| > |A_{12}| + |A_{13}| + \cdots + |A_{1n}|$$

$$|A_{22}| > |A_{21}| + |A_{23}| + \cdots + |A_{2n}|$$

$$|A_{33}| > |A_{31}| + |A_{32}| + |A_{34}| + \cdots + |A_{3n}|$$

⋮

$$|A_{nn}| > |A_{n1}| + |A_{n2}| + \cdots + |A_{n,n-1}|$$

$$|A_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^n |A_{ij}| \quad (i = 1, 2, \dots, n)$$

3-by-3 case

A is diagonally dominant if

$$|A_{11}| > |A_{12}| + |A_{13}|$$

$$|A_{22}| > |A_{21}| + |A_{23}|$$

$$|A_{33}| > |A_{31}| + |A_{32}|$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Example

Are the following matrices diagonally dominant?

$$\begin{bmatrix} -2 & 4 & -1 \\ 1 & -1 & 3 \\ 4 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Example

$$\begin{bmatrix} -2 & 4 & -1 \\ 1 & -1 & 3 \\ 4 & -2 & 1 \end{bmatrix}$$

Not diagonally dominant

$$\begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Diagonally dominant

Comparison of Different Methods

Limitations:

- Gaussian elimination and LU decomposition can solve any system of linear equations.
- Gauss-Seidel method does not work for all systems.

Accuracy:

- Gauss-Seidel method is more accurate than Gaussian elimination.
- Gaussian-elimination can produce large round-off errors.

Comparison of Different Methods

Efficiency:

- In the LU decomposition, you don't need to recompute L and U if A does not change.
- Gauss-Seidel method approximately requires $2n^2$ operations per iteration.
- Gaussian elimination method approximately requires $\frac{2n^3}{3}$ operations.
- Gauss-Seidel is more efficient than the Gaussian elimination when the number of iterations is less than or equal to $\frac{n}{3}$.

Eigenvalues and Eigenvectors

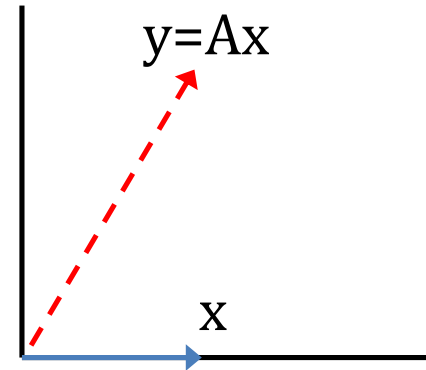
Eigenvalues and Eigenvectors

Matrix multiplication **often** changes the **direction** of a vector.

Example:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$
$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$y = Ax = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



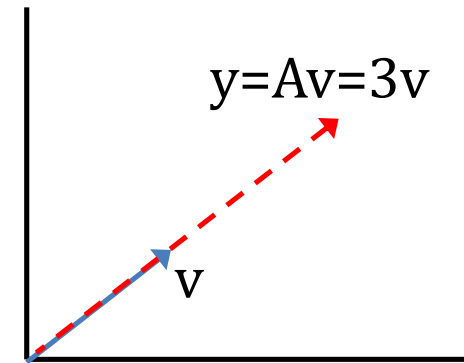
Multiplication by matrix A changes the direction of x.

Eigenvectors

- An **eigenvector** is a **vector**!
- However, **its direction** is **not** changed by **multiplication**.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$
$$v = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$$

$$y = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} = 3 \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$$



Multiplication does not change the direction of v . It only stretches v .

- v is an **eigenvector** of matrix A . **3** is the **eigenvalue associated** with v .

Motivation: PageRank Algorithm

- It was developed by **Google's founders**, Larry Page and Sergey Brin.
- It is used in most of **search engines** to **rank** the webpages.
- It worth **billions of dollars**.
- To rank webpages, it computes the **eigenvector** of a **certain** matrix.



Page and Brin

Motivation: PageRank Algorithm

- Rank of a webpage indicates its importance.
- Google's search engine displays the search results according to their ranks.
- More important result are displayed on the top of the list.

Google

 All

 Images

 Maps

 News

 Videos

 More

Settings

Tools

PageRank Algorithm



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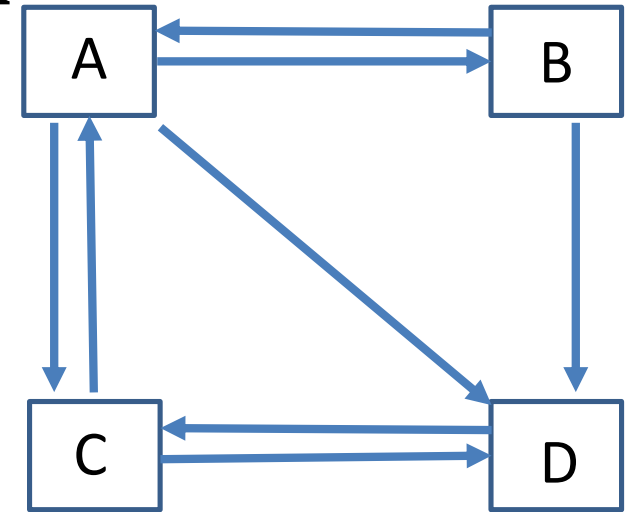
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PageRank Algorithm: Example

- Suppose you would like to rank webpages A, B, C, D.
- Rank of a webpage will a number between zero and one.
- r_A, r_B, r_C, r_D are the ranks of A, B, C, D, respectively.
- r_A measures the importance of A. Large values of r_A indicate that A is important.
- Sum of all ranks should be one (a simple normalization).



$$r_A + r_B + r_C + r_D = 1.$$

PageRank Algorithm: Example

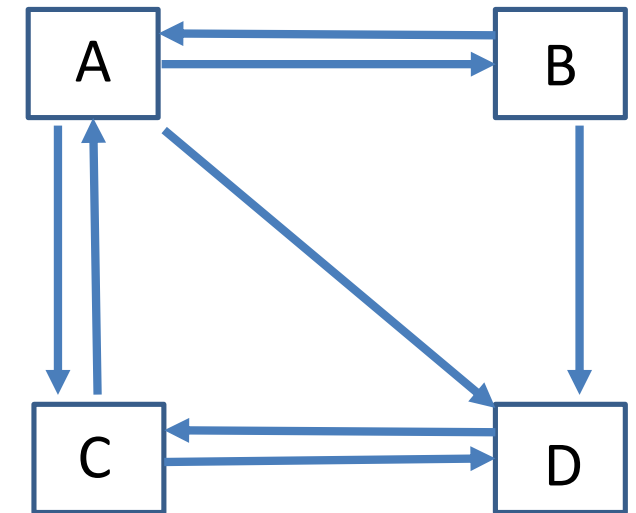
- Rank of a website depends on the rank of the websites that link to it.
- If an important website refers to A, it improves the rank of A. (Think of a social network)

$$r_A = \frac{1}{2}r_B + \frac{1}{2}r_C$$

$$r_B = \frac{1}{3}r_A$$

$$r_C = \frac{1}{3}r_A + r_D$$

$$r_D = \frac{1}{3}r_A + \frac{1}{2}r_B + \frac{1}{2}r_C$$



PageRank Algorithm: Example

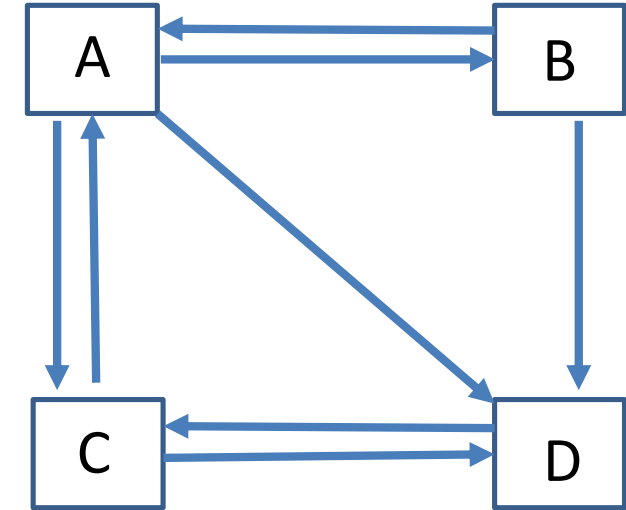
$$r_A = \frac{1}{2}r_B + \frac{1}{2}r_C$$

$$r_B = \frac{1}{3}r_A$$

$$r_C = \frac{1}{3}r_A + r_D$$

$$r_D = \frac{1}{3}r_A + \frac{1}{2}r_B + \frac{1}{2}r_C$$

$$r = \begin{bmatrix} r_A \\ r_B \\ r_C \\ r_D \end{bmatrix}$$



- Writing these equation in the matrix form, we have

$$r = Lr$$

- Multiplication** by matrix **L** **does not change** the direction of the rank vector.
- Rank vector** is an **eigenvector** of the matrix **L** (Link matrix).

$$L = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Link matrix

Page Rank Algorithm: Example

- Solution:

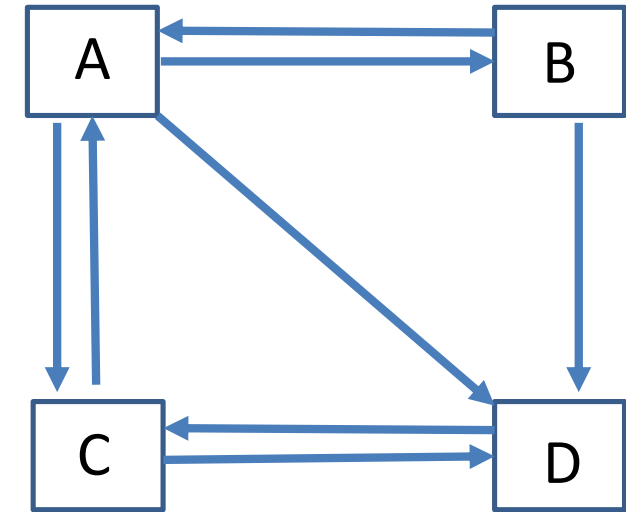
$$r_A = 0.23$$

$$r_B = 0.09$$

$$r_C = 0.37$$

$$r_D = 0.31$$

$$\mathbf{r} = \mathbf{Lr}$$



- C is the most important website!
- B is the least important one!
- Google ranks the websites by computing the eigenvector of $\mathbf{L}$.
- In 2002, the size of link matrix was 2.7 billion by 2.7 billion!

$$\mathbf{L} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Link matrix

Eigenvalues and Eigenvectors

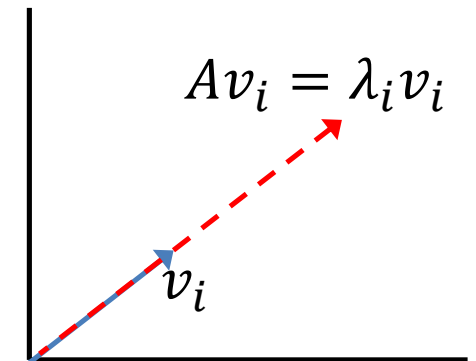
- An $n \times n$ matrix has n eigenvalues.
- Eigenvalues and eigenvectors form pairs.
- Each eigenvalue has a corresponding eigenvector.

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

$$Av_i = \lambda_i v_i \quad i = 1, \dots, n$$

$$\begin{array}{cccc} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \downarrow & \downarrow & & \downarrow \\ v_1 & v_2 & \dots & v_n \end{array}$$

- λ_i determines how much v_i scales.



Eigenvalues and Eigenvectors

In this lecture, you will learn how to **compute** the **eigenvalues** and **eigenvectors**!

Eigenvalues

How to compute eigenvalues of A?

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

- Step1: Form the matrix $A - \lambda I$ (λ is a variable!)
- I is an n-by-n **diagonal** matrix with all entries equal to one.
- I is called an identity matrix.

$$I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

$$\lambda I = \begin{bmatrix} \lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda \end{bmatrix}$$

Diagonal matrix with all
diagonal elements equal to λ

$$A - \lambda I = \begin{bmatrix} A_{11} - \lambda & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} - \lambda & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} - \lambda & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} - \lambda \end{bmatrix}$$

Eigenvalues

- Step2: Find the **characteristic polynomial** of A

$$\det(A - \lambda I) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n$$

Determinant of $A - \lambda I$

Characteristic polynomial of A

$$A - \lambda I = \begin{bmatrix} A_{11} - \lambda & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} - \lambda & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} - \lambda & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} - \lambda \end{bmatrix}$$

Eigenvalues

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

3. Find the **roots** of the **characteristic polynomial**.

$$a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n = 0$$

This equation has n solutions:

$$\lambda_1, \dots, \lambda_n$$

Eigenvalues are the **roots** of the **characteristic polynomial**.

Example

Find the eigenvalues of matrix A.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

Example

Step 1 Form $A - \lambda I$

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 - \lambda & 2 \\ 1 & 2 - \lambda \end{bmatrix}$$

Step 2 Find the characteristic polynomial

$$\begin{aligned} \det(A - \lambda I) &= \det \left(\begin{bmatrix} 3 - \lambda & 2 \\ 1 & 2 - \lambda \end{bmatrix} \right) \\ &= (3 - \lambda)(2 - \lambda) - 2 \\ &= \lambda^2 - 5\lambda + 4 \end{aligned}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

Example

Step 3 Find the roots of the characteristic polynomial

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = 4$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

Find the eigenvalues of matrix A.

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

Example

Step 1 Form $A - \lambda I$

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 - \lambda & 2 & 2 \\ 2 & 2 - \lambda & 0 \\ 2 & 0 & 4 - \lambda \end{bmatrix}$$

Example

Step 2 Find the characteristic polynomial

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \det \left(\begin{bmatrix} 3 - \lambda & 2 & 2 \\ 2 & 2 - \lambda & 0 \\ 2 & 0 & 4 - \lambda \end{bmatrix} \right) \\ &= \lambda^3 - 9\lambda^2 + 18\lambda \end{aligned}$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Example

Step 3 Find the roots of the characteristic polynomial

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

$$\begin{aligned}\lambda^3 - 9\lambda^2 + 18\lambda &= 0 \\ \lambda(\lambda^2 - 9\lambda + 18) &= 0\end{aligned}$$

$$\lambda_1 = 0$$

$$\lambda_2 = 3$$

$$\lambda_2 = 6$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

END

See you next week 😊!