

MA1200 CALCULUS AND BASIC LINEAR ALGEBRA

LECTURE: CG1

REVIEW EXAMPLES ON CHAPTER 6 TO 8

Example 1 (Exam 1617B)

Evaluate the following limits:

$$(a) \quad \lim_{x \rightarrow 0} \frac{1 - \cos(3x)}{x^2}$$

$$(b) \quad \lim_{x \rightarrow \infty} \frac{1 - x^2}{1 + x^2}$$

$$(c) \quad \lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{3}{3x + 2x^2} \right)$$

Solutions to Review Examples on Ch. 6-8

Example 1:

(a) Method 1:

$$\lim_{x \rightarrow 0} \frac{1 - \cos(3x)}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2(\frac{3x}{2})}{x^2}, \text{ using Half-angle formula:}$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta) \text{ where } \theta = \frac{3x}{2}$$

$$= \lim_{x \rightarrow 0} 2 \cdot \frac{\sin^2(\frac{3x}{2})}{(\frac{3x}{2})^2} \cdot (\frac{3}{2})^2$$

$$= 2 \cdot 1^2 \cdot \frac{9}{4}$$

$$= \frac{9}{2}$$

Method 2:

$$\lim_{x \rightarrow 0} \frac{1 - \cos(3x)}{x^2} \text{ (}\frac{0}{0}\text{ form)}$$

$$= \lim_{x \rightarrow 0} \frac{3 \sin(3x)}{2x}, \text{ by L'Hôpital's Rule (}\frac{0}{0}\text{ form)}$$

$$= \lim_{x \rightarrow 0} \frac{3 \cdot 3 \cos(3x)}{2}, \text{ by L'Hopital's rule}$$

$$= \frac{9}{2} \cdot \cos 0$$

$$= \frac{9}{2}$$

(b) Method 1:

$$\lim_{x \rightarrow \infty} \frac{1-x^2}{1+x^2} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} + 1} = \frac{0-1}{0+1} = -1$$

Method 2:

$$\lim_{x \rightarrow \infty} \frac{1-x^2}{1+x^2} \quad \left(\frac{-\infty}{\infty} \text{ form} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{-2x}{2x}, \text{ by L'Hôpital's rule}$$

$$= \lim_{x \rightarrow \infty} (-1)$$

$$= -1$$

$$(C) \quad \lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{3}{3x + 2x^2} \right) \quad (\infty - \infty \text{ form})$$

$$= \lim_{x \rightarrow 0^+} \frac{(3x + 2x^2) - 3x}{x(3x + 2x^2)}$$

$$= \lim_{x \rightarrow 0^+} \frac{2x^2}{x^2(3 + 2x)}$$

$$= \lim_{x \rightarrow 0^+} \frac{2}{3 + 2x}$$

$$= \frac{2}{3 + 0}$$

$$= \frac{2}{3}$$

Example 2 (Exam 1314B)

$$\text{Let } f(x) = \begin{cases} \frac{x^3 - 8}{x - 2} & \text{if } x \neq 2 \\ c & \text{if } x = 2 \end{cases}.$$

Find the value of c for which $f(x)$ is continuous at $x = 2$. Give your reason.

Example 2

$$f(x) = \begin{cases} \frac{x^3-8}{x-2} & \text{if } x \neq 2 \\ c & \text{if } x = 2 \end{cases}$$

$$\lim_{x \rightarrow 2} \frac{x^3-8}{x-2} \quad \left(\frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(x^2+2x+4)}{x-2}$$

$$= \lim_{x \rightarrow 2} (x^2+2x+4)$$

$$= 2^2 + 2(2) + 4$$

$$= 12$$

$$f(x) \text{ is continuous at } x=2 \text{ iff } \lim_{x \rightarrow 2} f(x) = f(2) \\ = c,$$

$$\text{i.e. } c = 12.$$

Example 3 (Exam 1213A)

Let $g(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Determine whether $g(x)$ is differentiable at $x = 0$, if so, find the value of the first derivative there.

Example 3

$$g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$g(x)$ is differentiable at $x=0$ iff $\lim_{x \rightarrow 0} \frac{g(x)-g(0)}{x-0}$ exists.

$$\lim_{x \rightarrow 0} \frac{g(x)-g(0)}{x-0} = \lim_{x \rightarrow 0} \frac{x^2 \sin\left(\frac{1}{x}\right) - 0}{x} = \lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right).$$

Since $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for all $x \neq 0$, we have

$$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x| \quad \text{for all } x \neq 0.$$

$$\lim_{x \rightarrow 0} (|x|) = 0 = \lim_{x \rightarrow 0} |x|$$

$\therefore$ By the Sandwich Theorem, $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$.

$\therefore \lim_{x \rightarrow 0} \frac{g(x)-g(0)}{x-0}$ exists and $\lim_{x \rightarrow 0} \frac{g(x)-g(0)}{x-0} = \lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$

$\therefore g(x)$ is differentiable at $x=0$ and $g'(0) = 0$.

Example 4 (Exam 1617B)

(a) Prove from first principles that $\frac{d}{dx}(x^3) = 3x^2$.

(b) Let $F(x) = |\cos x|$, for $x \in \mathbf{R}$. Determine whether $F(x)$ is differentiable at $x = 0$. Give your reason.

(Hint: You may use $\cos 2\theta = 1 - 2\sin^2 \theta$.)

Example 4

(a) Let $f(x) = x^3$.

From the First Principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

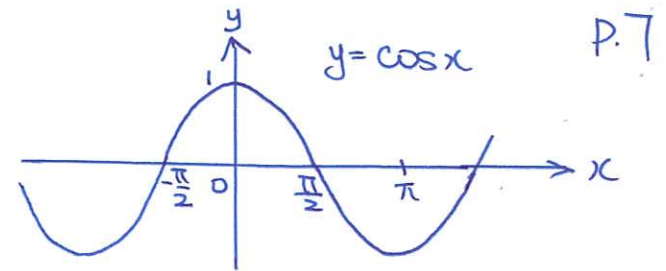
$$= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2)$$

$$= 3x^2 + 3x \cdot 0 + 0^2$$

$$= 3x^2$$

$$\therefore \frac{d}{dx}(x^3) = 3x^2$$

$$\begin{aligned}
 (b) \quad F(x) &= |\cos x| = \begin{cases} \cos x & \text{if } \cos x \geq 0 \\ -\cos x & \text{if } \cos x < 0 \end{cases} \\
 &= \cos x \quad \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}
 \end{aligned}$$



$$\lim_{x \rightarrow 0} \frac{F(x) - F(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\cos x - |\cos 0|}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\cos x - 1}{x}$$

$$= \lim_{x \rightarrow 0} \frac{-2 \sin^2(\frac{x}{2})}{x}$$

$$= \lim_{x \rightarrow 0} -\sin(\frac{x}{2}) \cdot \frac{\sin(\frac{x}{2})}{\frac{x}{2}}$$

$$= -\sin(0) \cdot 1$$

$$= 0$$

$$\therefore \lim_{x \rightarrow 0} \frac{F(x) - F(0)}{x - 0} \text{ exists}$$

$\therefore F(x)$ is differentiable at $x=0$.

$$\because \cos 2\theta = 1 - 2 \sin^2 \theta$$

$$\Rightarrow \cos 2\theta - 1 = -2 \sin^2 \theta$$

$$(\text{Put } \theta = \frac{x}{2})$$

Example 5

Find $\frac{dy}{dx}$ for each of the following:

(a) $y = \tan^{-1}\left[5 \cosh(x^3 + 2)\right]$

(b) $y = \left(\sqrt{x} + \frac{1}{x}\right)^{x^2}$

(c) $xy^2 + e^{-xy} = 1$

(d) $y = \cos^3(4x) \cdot \ln[\sinh(2x)]$

(e) $y = \frac{(2x+3)^6}{(x^2+1)e^{4x}}$

Example 5

P.8

$$(a) \quad y = \tan^{-1}[5 \cosh(x^3+2)]$$

$$\frac{dy}{dx} = \frac{1}{1 + [5 \cosh(x^3+2)]^2} \cdot 5 \sinh(x^3+2) \cdot 3x^2$$

$$= \frac{15x^2 \sinh(x^3+2)}{1 + 25 \cosh^2(x^3+2)}$$

$$(b) \quad y = \left(\sqrt{x} + \frac{1}{x}\right)^{x^2}$$

Take $\ln$ on both sides:

$$\ln y = \ln\left[\left(\sqrt{x} + \frac{1}{x}\right)^{x^2}\right] = x^2 \ln(x^{\frac{1}{2}} + x^{-1})$$

Differentiate both sides w.r.t. x :

$$\frac{1}{y} \frac{dy}{dx} = x^2 \cdot \frac{1}{x^{\frac{1}{2}} + x^{-1}} \cdot \left(\frac{1}{2}x^{-\frac{1}{2}} - x^{-2}\right) + 2x \ln(x^{\frac{1}{2}} + x^{-1})$$

$$\Rightarrow \frac{dy}{dx} = \left(\sqrt{x} + \frac{1}{x}\right)^{x^2} \left[\frac{x^2}{\sqrt{x} + \frac{1}{x}} \left(\frac{1}{2\sqrt{x}} - \frac{1}{x^2} \right) + 2x \ln\left(\sqrt{x} + \frac{1}{x}\right) \right]$$

$$(c) \quad xy^2 + e^{-xy} = 1$$

Differentiate both sides w.r.t. x :

$$\left[x \cdot 2y \frac{dy}{dx} + y^2 \right] + e^{-xy} (-x \frac{dy}{dx} - y) = 0$$

$$\Rightarrow (2xy - xe^{-xy}) \frac{dy}{dx} = ye^{-xy} - y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{ye^{-xy} - y^2}{2xy - xe^{-xy}}$$

$$(d) \quad y = \cos^3(4x) \cdot \ln[\sinh(2x)]$$

$$\frac{dy}{dx} = \cos^3(4x) \cdot \frac{1}{\sinh(2x)} \cdot \cosh(2x) \cdot 2 + \ln[\sinh(2x)] \cdot 3\cos^2(4x) \cdot [-\sin(4x)] \cdot 4$$

$$= 2\cos^3(4x) \cdot \coth(2x) - 12\cos^2(4x) \cdot \sin(4x) \cdot \ln[\sinh(2x)]$$

$$(e) \quad y = \frac{(2x+3)^6}{(x^2+1)e^{4x}}$$

Take $\ln$ on both sides :

$$\begin{aligned}\ln y &= \ln \left[\frac{(2x+3)^6}{(x^2+1)e^{4x}} \right] \\ &= 6 \ln(2x+3) - \ln(x^2+1) - 4x\end{aligned}$$

Differentiate both sides w.r.t. x :

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= 6 \cdot \frac{1}{2x+3} \cdot 2 - \frac{2x}{x^2+1} - 4 \\ \Rightarrow \frac{dy}{dx} &= \frac{(2x+3)^6}{(x^2+1)e^{4x}} \left[\frac{12}{2x+3} - \frac{2x}{x^2+1} - 4 \right]\end{aligned}$$

Example 6 (Exam 1415A)

A curve has parametric equations

$$x = 2t - \log_e(2t),$$

$$y = t^2 - \log_e(t^2),$$

where t is the parameter and $t > 0$.

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t .

Example 6

$$x = 2t - \ln(2t) \Rightarrow \frac{dx}{dt} = 2 - \frac{2}{2t} = 2 - \frac{1}{t}$$

$$y = t^2 - \ln(t^2) \Rightarrow \frac{dy}{dt} = 2t - \frac{2t}{t^2} = 2t - \frac{2}{t}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - \frac{2}{t}}{2 - \frac{1}{t}} = \frac{2t^2 - 2}{2t - 1}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{2t^2 - 2}{2t - 1}\right)}{2 - \frac{1}{t}}$$

$$= \frac{\frac{(2t-1)(4t) - (2t^2-2)(2)}{(2t-1)^2}}{2 - \frac{1}{t}}$$

$$= \frac{t(4t^2 - 4t + 4)}{(2t-1)^3}$$

Example 7 (Exam 1415A)

Given that $y = e^{\sqrt{3}x} \cos x$, express $\frac{dy}{dx}$ in the form $re^{\sqrt{3}x} \cos(x + \phi)$, where

$r > 0$ and $0 < \phi < \frac{\pi}{2}$, and state the numerical value of r and ϕ . Express $\frac{d^2y}{dx^2}$

in similar form. (Hint: $\cos(A + B) = \cos A \cos B - \sin A \sin B$.)

Example 7

$$y = e^{\sqrt{3}x} \cos x$$

$$\frac{dy}{dx} = e^{\sqrt{3}x} (-\sin x) + \cos x \cdot \sqrt{3} e^{\sqrt{3}x}$$

$$= e^{\sqrt{3}x} (\sqrt{3} \cos x - \sin x)$$

$$\equiv r e^{\sqrt{3}x} \cos(x + \phi) = r e^{\sqrt{3}x} (\cos x \cos \phi - \sin x \sin \phi)$$

$$\therefore \begin{cases} r \cos \phi = \sqrt{3} \\ r \sin \phi = 1 \end{cases}$$

$$\Rightarrow (r \cos \phi)^2 + (r \sin \phi)^2 = (\sqrt{3})^2 + 1^2$$

$$\Rightarrow r^2 (\cos^2 \phi + \sin^2 \phi) = 4$$

$$\Rightarrow r^2 = 4$$

$$\Rightarrow r = 2 \quad \text{or} \quad -2 \quad (\text{rejected} \because r > 0)$$

$$\tan \phi = \frac{\sin \phi}{\cos \phi} = \frac{1}{\sqrt{3}} \Rightarrow \phi = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\therefore \frac{dy}{dx} = 2 e^{\sqrt{3}x} \cos\left(x + \frac{\pi}{6}\right)$$

From above,

$$\sqrt{3} \cos x - \sin x = 2 \cos(x + \frac{\pi}{6}) \quad \text{--- } (*)$$

Then

$$\begin{aligned} \frac{d^2 y}{dx^2} &= 2 \frac{d}{dx} [e^{\sqrt{3}x} \cos(x + \frac{\pi}{6})] \\ &= 2 \cdot e^{\sqrt{3}x} [\sqrt{3} \cos(x + \frac{\pi}{6}) - \sin(x + \frac{\pi}{6})] \\ &= 2 e^{\sqrt{3}x} [2 \cos(x + \frac{\pi}{6} + \frac{\pi}{6})] \quad \text{by } (*) \\ &= 2^2 e^{\sqrt{3}x} \cos(x + \frac{\pi}{3}) \end{aligned}$$

Example 8 (Exam 1415B)

- (a) Express $\frac{7x+17}{(x-4)(2x+1)}$ in partial fractions.
- (b) If $y = (\alpha x + \beta)^{-1}$, where α and β are non-zero constants, find the general formula for the n th derivative of y with respect to x .
- (c) Using the result in parts (a) and (b), or otherwise, find the sixth derivative of $\frac{7x+17}{(x-4)(2x+1)}$ with respect to x . You need not simplify your answer.

Example 8

P. 14

$$(a) \quad \frac{7x+17}{(x-4)(2x+1)} = \frac{A}{x-4} + \frac{B}{2x+1}$$

$$\Rightarrow 7x+17 = A(2x+1) + B(x-4)$$

$$\text{Put } x=4 : 45 = 9A \Rightarrow A=5$$

$$\text{Put } x=-\frac{1}{2} : \frac{27}{2} = -\frac{9B}{2} \Rightarrow B=-3$$

$$\therefore \frac{7x+17}{(x-4)(2x+1)} = \frac{5}{x-4} - \frac{3}{2x+1}$$

$$(b) \quad y = (\alpha x + \beta)^{-1}$$

$$\frac{dy}{dx} = -(\alpha x + \beta)^{-2} \cdot \alpha = -\alpha(\alpha x + \beta)^{-2}$$

$$\frac{d^2y}{dx^2} = -\alpha(-2)(\alpha x + \beta)^{-3} \cdot \alpha = (-1)^2 \cdot 2! \alpha^2 (\alpha x + \beta)^{-3}$$

$$\frac{d^3y}{dx^3} = (-1)^2 \cdot 2! \alpha^2 (-3)(\alpha x + \beta)^{-4} \cdot \alpha = (-1)^3 \cdot 3! \alpha^3 (\alpha x + \beta)^{-4}$$

$$\therefore \text{By conjecture, } \frac{d^n y}{dx^n} = (-1)^n \cdot n! \alpha^n (\alpha x + \beta)^{-(n+1)}, \text{ for } n \geq 1.$$

$$\begin{aligned} (c) \quad \frac{d^6}{dx^6} \left(\frac{7x+17}{(x-4)(2x+1)} \right) &= 5 \cdot \frac{d^6}{dx^6} [(x-4)^{-1}] - 3 \cdot \frac{d^6}{dx^6} [(2x+1)^{-1}] \quad , \text{ from (a)} \\ &= 5 \cdot (-1)^6 \cdot 6! \cdot 1^6 \cdot (x-4)^{-7} - 3 \cdot (-1)^6 \cdot 6! \cdot 2^6 (2x+1)^{-7} \quad , \text{ from (b)} \\ &= 5 \cdot 6! \cdot (x-4)^{-7} - 3 \cdot 6! \cdot 2^6 (2x+1)^{-7} \end{aligned}$$

Example 9 (Exam 1617B)

Find the seventh derivative of $(x^2 - x + 3)e^{-2x}$ with respect to x .

Example 9 :

P.16

By the Leibnitz rule,

$$\frac{d^7}{dx^7} [(x^2 - x + 3) e^{-2x}]$$

$$= \sum_{k=0}^7 \binom{7}{k} (x^2 - x + 3)^{(k)} (e^{-2x})^{(7-k)}$$

all the remaining
terms are 0

$$= \binom{7}{0} (x^2 - x + 3)^{(0)} (e^{-2x})^{(7)} + \binom{7}{1} (x^2 - x + 3)^{(1)} (e^{-2x})^{(6)} + \binom{7}{2} (x^2 - x + 3)^{(2)} (e^{-2x})^{(5)} + 0$$

$$= 1 \cdot (x^2 - x + 3) \cdot (-2)^7 e^{-2x} + 7(2x - 1) \cdot (-2)^6 e^{-2x} + \frac{7(6)}{2} \cdot 2 \cdot (-2)^5 e^{-2x}$$

$$\left[\because \frac{d^n}{dx^n} (e^{ax}) = a^n e^{ax} \right]$$

$$= e^{-2x} \cdot (-2)^5 [4(x^2 - x + 3) - 2(7)(2x - 1) + 42]$$

$$= e^{-2x} \cdot (-2)^5 \cdot (4x^2 - 32x + 68)$$

Example 10 (Exam 1314B)

$P\left(2, \frac{2+\sqrt{3}}{2}\right)$ is a point on the curve $x^2 + 4y^2 - 6x - 8y + 9 = 0$.

- (a) Find the slope of the tangent to the curve at P .
- (b) Find the equation of the normal to the curve at P .

Example 10

$$(a) \quad x^2 + 4y^2 - 6x - 8y + 9 = 0$$

Differentiate both sides w.r.t. x :

$$2x + 8y \frac{dy}{dx} - 6 - 8 \frac{dy}{dx} = 0$$

$$\Rightarrow (8y - 8) \frac{dy}{dx} = 6 - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{6 - 2x}{8y - 8} = \frac{3 - x}{4y - 4}$$

At $P(2, \frac{2+\sqrt{3}}{2})$, $\frac{dy}{dx} = \frac{3 - 2}{4(\frac{2+\sqrt{3}}{2}) - 4} = \frac{1}{2\sqrt{3}}$, which is the slope of the tangent to the curve at P .

(b) Equation of the normal to the curve at P is

$$\frac{y - \frac{2+\sqrt{3}}{2}}{x-2} = \frac{-1}{\left(\frac{1}{2\sqrt{3}}\right)}$$

$$\Rightarrow y = -2\sqrt{3}(x-2) + \frac{2+\sqrt{3}}{2}$$

$$= -2\sqrt{3}x + 1 + \frac{9\sqrt{3}}{2}$$

Example 11

Evaluate $\lim_{x \rightarrow 0} \frac{1 + x - e^{\sin x}}{x - \ln(1 + x)}.$

Example 11

$$\lim_{x \rightarrow 0} \frac{1+x-e^{\sin x}}{x-\ln(1+x)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x \cdot e^{\sin x}}{1 - \frac{1}{1+x}}, \quad \text{by L'Hôpital's rule, } \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \rightarrow 0} \frac{-\cos x \cdot e^{\sin x} \cdot \cos x - e^{\sin x} (-\sin x)}{\frac{1}{(1+x)^2}}, \quad \text{by L'Hopital's rule}$$

$$= \frac{-\cos^2 0 \cdot e^{\sin 0} + \sin 0 \cdot e^{\sin 0}}{\frac{1}{(1+0)^2}}$$

$$= \frac{-1 + 0}{1}$$

$$= -1$$

Example 12 (Exam 1415B)

(a) Starting from the formula $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, show that

$$\tan 4\theta = \frac{4t(1-t^2)}{1-6t^2+t^4}, \text{ where } t = \tan \theta. \text{ Deduce that, when } \theta = \tan^{-1}\left(\frac{1}{5}\right),$$

$$\tan\left(4\theta - \frac{\pi}{4}\right) = \frac{1}{239}.$$

(b) If $y = \tan^{-1} x$, prove that $(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$. By repeated differentiation

of this result and use the Maclaurin theorem, or otherwise, prove that the first three

non-zero terms in the series expansion of $\tan^{-1} x$ are $x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$.

(c) Using the results in parts (a) and (b), find an approximation to the value of π , giving 5 decimal places in your answer.

Example 12

(a) Given that $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ ———— (*)

Then $\tan 2\theta = \tan(\theta + \theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2t}{1 - t^2}$, where $t = \tan \theta$.

Then $\tan 4\theta = \tan(2\theta + 2\theta) = \frac{2 \tan(2\theta)}{1 - \tan^2(2\theta)} = \frac{2 \left(\frac{2t}{1-t^2} \right)}{1 - \left(\frac{2t}{1-t^2} \right)^2} = \frac{\frac{4t}{1-t^2}}{\frac{(1-t^2)^2 - 4t^2}{(1-t^2)^2}}$

$$= \frac{4t(1-t^2)}{1 - 6t^2 + t^4},$$

Where $t = \tan \theta$.

When $\theta = \tan^{-1}(\frac{1}{5})$, $t = \tan \theta = \frac{1}{5}$. Then

$$\tan 4\theta = \frac{4(\frac{1}{5})[1 - (\frac{1}{5})^2]}{1 - 6(\frac{1}{5})^2 + (\frac{1}{5})^4} = \frac{\frac{96}{125}}{\frac{476}{625}} = \frac{120}{119}$$

and $\tan(4\theta - \frac{\pi}{4}) = \frac{\tan(4\theta) + \tan(-\frac{\pi}{4})}{1 - \tan(4\theta) \tan(-\frac{\pi}{4})}$ by (*)

$$= \frac{\frac{120}{119} - 1}{1 - \frac{120}{119} \cdot (-1)} = \frac{1}{239}$$

(b) $y = \tan^{-1} x$

$$\frac{dy}{dx} = \frac{1}{1+x^2} \Rightarrow (1+x^2) \frac{dy}{dx} = 1$$

Differentiate both sides w.r.t. x :

$$(1+x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 0 \quad \text{--- ①}$$

Differentiate both sides of ① w.r.t. x :

$$(1+x^2) y^{(3)} + 2x y'' + 2x y'' + 2 y' = 0$$

$$\Rightarrow (1+x^2) y^{(3)} + 4x y'' + 2 y' = 0 \quad \text{--- ②}$$

Differentiate both sides of ② w.r.t. x :

$$(1+x^2) y^{(4)} + 2x y^{(3)} + 4x y^{(3)} + 4 y'' + 2 y'' = 0$$

$$\Rightarrow (1+x^2) y^{(4)} + 6x y^{(3)} + 6 y'' = 0 \quad \text{--- ③}$$

Differentiate both sides of ③ w.r.t. x :

$$(1+x^2) y^{(5)} + 2x y^{(4)} + 6x y^{(4)} + 6 y^{(3)} + 6 y^{(3)} = 0$$

$$\Rightarrow (1+x^2) y^{(5)} + 8x y^{(4)} + 12 y^{(3)} = 0$$

Putting $x=0$ into the above results:

$$y(0) = \tan^{-1} 0 = 0$$

$$y'(0) = \frac{1}{1+0^2} = 1$$

$$y''(0) = -2 \cdot 0 \cdot y'(0) = 0, \text{ from ①.}$$

$$y^{(3)}(0) = \underbrace{-4 \cdot 0 \cdot y''(0)}_{=0} - 2 \underbrace{y'(0)}_{=1} = -2y'(0) = -2, \text{ from ②}$$

$$y^{(4)}(0) = -6 \underbrace{y''(0)}_{=0} = 0, \text{ from ③}$$

$$y^{(5)}(0) = -12 \underbrace{y^{(3)}(0)}_{=-2} = 24, \quad \text{from (4)}$$

$\therefore$ Maclaurin series of $\tan^{-1}x$ is

$$y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

$$= 0 + x + 0 + \frac{-2}{3!}x^3 + 0 + \frac{24}{5!}x^5 + \dots$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots$$

(c) From (a), when $\theta = \tan^{-1}(\frac{1}{5})$, $\tan(4\theta - \frac{\pi}{4}) = \frac{1}{239}$.

$$\therefore 4 \tan^{-1}(\frac{1}{5}) - \frac{\pi}{4} = \tan^{-1}(\frac{1}{239})$$

$$\Rightarrow \pi = 4 \left[4 \tan^{-1}(\frac{1}{5}) - \tan^{-1}(\frac{1}{239}) \right]$$

$$= 4 \left\{ 4 \left[\frac{1}{5} - \frac{1}{3} \left(\frac{1}{5} \right)^3 + \frac{1}{5} \left(\frac{1}{5} \right)^5 - \dots \right] - \left[\frac{1}{239} - \frac{1}{3} \left(\frac{1}{239} \right)^3 + \frac{1}{5} \left(\frac{1}{239} \right)^5 - \dots \right] \right\}, \text{ using (b)}$$

$$\approx 3.14162 \text{ (to 5 d.p.)}$$

Example 13

Let $P(0, -2)$ be a turning point of the curve $y = \frac{x^2 + px + q}{x + 1}$.

- (a) Find the values of p and q .
- (b) Find all local extrema of the curve.
- (c) Find the largest possible domain and largest possible range of the function

$$f(x) = \frac{x^2 + px + q}{x + 1}, \text{ where } p \text{ and } q \text{ are the values you found in (a).}$$

Example 13

$$(a) \quad y = \frac{x^2 + px + q}{x+1}$$

$$y' = \frac{(x+1)(2x+p) - (x^2+px+q)(1)}{(x+1)^2} = \frac{x^2 + 2x + p - q}{(x+1)^2}$$

$\therefore P(0, -2)$ is a turning point of the curve.

$$\therefore y(0) = -2 \quad \& \quad y'(0) = 0$$

$$\text{i.e.} \quad -2 = \frac{0+0+q}{0+1} \Rightarrow q = -2$$

$$\& \quad 0 = \frac{p-q}{1^2} \Rightarrow p = q = -2$$

$$(b) \quad y = \frac{x^2 - 2x - 2}{x+1}$$

$$y' = \frac{x^2 + 2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$$

$$\text{Set } y' = 0 \Rightarrow \frac{x(x+2)}{(x+1)^2} = 0 \Rightarrow x(x+2) = 0 \Rightarrow x = 0 \text{ or } x = -2$$

	$x < -2$	$x = -2$	$-2 < x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$x > 0$
sign of x	-	-	-	~~X~~	-	0	+
sign of $(x+2)$	-	0	+	~~X~~	+	+	+
sign of y'	+	0	-	~~X~~	-	0	+

$\cap$
 $\cup$

By the First derivative test, the curve has
 a local maximum at $x = -2$, $y = -6$
 & a local minimum at $x = 0$, $y = -2$.

(c) $f(x) = \frac{x^2 - 2x - 2}{x+1}$ is defined when $x+1 \neq 0 \Rightarrow x \neq -1$.

$\therefore \text{Dom}(f) = \mathbb{R} \setminus \{-1\}$

Let $y = \frac{x^2 - 2x - 2}{x+1}$. Then $(x+1)y = x^2 - 2x - 2$

$\Rightarrow x^2 + (-2-y)x - 2-y = 0$

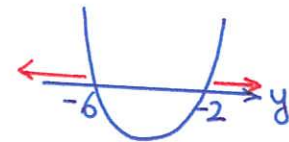
$\Rightarrow x = \frac{-(-2-y) \pm \sqrt{(-2-y)^2 - 4(1)(-2-y)}}{2(1)}$

Which has real solutions when

$(-2-y)^2 - 4(1)(-2-y) \geq 0 \Rightarrow y^2 + 8y + 12 \geq 0$

$\Rightarrow (y+6)(y+2) \geq 0$

$\Rightarrow y \leq -6 \text{ or } y \geq -2$



$\therefore \text{Ran}(f) = (-\infty, -6] \cup [-2, \infty)$

Example 14

Let $y = \sinh^{-1} x$. Use implicit differentiation to show that $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$.

Hence, or otherwise show that

$$(1+x^2)y^{(2)} + xy^{(1)} = 0,$$

and that, for $n \geq 0$,

$$(1+x^2)y^{(n+2)} + (2n+1)xy^{(n+1)} + n^2y^{(n)} = 0,$$

where $y^{(r)}$ denotes $\frac{d^r y}{dx^r}$.

Hence, or otherwise, find the expansion of $\sinh^{-1} x$ in ascending powers of x as far as the term in x^5 .

Example 14

$$y = \sinh^{-1} x \Rightarrow x = \sinh y$$

Differentiate both sides w.r.t. x :

$$1 = \cosh y \cdot \frac{dy}{dx}$$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} \\ &= \frac{1}{\sqrt{1 + x^2}} \end{aligned}$$

$$\because \cosh^2 y - \sinh^2 y = 1$$

$$\& \cosh y = \frac{1}{2}(e^y + e^{-y}) \geq 1$$

for all $y \in \mathbb{R}$.

$$\therefore (1+x^2)^{\frac{1}{2}} \cdot y' = 1$$

Differentiate both sides w.r.t. x :

$$(1+x^2)^{\frac{1}{2}} y'' + \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x y' = 0$$

Multiplying both sides by $(1+x^2)^{\frac{1}{2}}$:

$$(1+x^2) y'' + x y' = 0 \quad \text{————— ①}$$

using Leibnitz' rule and differentiating both sides of ① n times w.r.t. x : P.29

$$\left[\sum_{k=0}^n \binom{n}{k} (1+x^2)^{(k)} (y'')^{(n-k)} \right] + \left[\sum_{k=0}^n \binom{n}{k} x^{(k)} (y')^{(n-k)} \right] = 0^{(n)}$$

$$\Rightarrow \left[1 \cdot (1+x^2) y^{(n+2)} + n(2x) y^{(n+1)} + \frac{n(n-1)}{2} \cdot 2 \cdot y^{(n)} \right] \\ + \left[1 \cdot x \cdot y^{(n+1)} + n \cdot 1 \cdot y^{(n)} \right] = 0$$

$$\Rightarrow (1+x^2) y^{(n+2)} + (2n+1)x \cdot y^{(n+1)} + n^2 y^{(n)} = 0 \text{ ———— ②} \\ \text{for } n \geq 2.$$

If we differentiate both sides of ① w.r.t. x , we have

$$(1+x^2) y^{(3)} + 3x y'' + y' = 0 \text{ ———— ③}$$

Comparing ① and ③ with ②,

$$(1+x^2) y^{(n+2)} + (2n+1)x \cdot y^{(n+1)} + n^2 y^{(n)} = 0 \text{ ———— (*)}$$

is true for $n \geq 0$.

To find the expansion of $\sinh^{-1}x$ in ascending powers of x :

← Maclaurin series

Putting $x=0$ into $(*)$, we have

$$y^{(n+2)}(0) + n^2 y^{(n)}(0) = 0$$

$$\Rightarrow y^{(n+2)}(0) = -n^2 \cdot y^{(n)}(0) \quad \text{--- } (**), \text{ for } n \geq 0.$$

We have $y(0) = \sinh^{-1}0 = 0$ $(\because \sinh 0 = \frac{1}{2}(e^0 - e^{-0}) = 0)$

$$y'(0) = \frac{1}{\sqrt{1+0^2}} = 1$$

$$y''(0) = 0, \text{ using } (**) \text{ with } n=0$$

$$y^{(3)}(0) = -1^2 \cdot y'(0) = -1, \text{ using } (**) \text{ with } n=1$$

$$y^{(4)}(0) = -2^2 \cdot y''(0) = 0, \text{ using } (**) \text{ with } n=2$$

$$y^{(5)}(0) = -3^2 \cdot y^{(3)}(0) = -9 \cdot (-1) = 9, \text{ using } (**) \text{ with } n=3$$

$$\therefore \sinh^{-1} x = y(0) + \frac{y'(0)}{1!} x + \frac{y''(0)}{2!} x^2 + \frac{y^{(3)}(0)}{3!} x^3 + \dots$$

$$= 0 + \frac{1}{1!} x + \frac{0}{2!} x^2 + \frac{(-1)}{3!} x^3 + \frac{0}{4!} x^4 + \frac{9}{5!} x^5 + \dots$$

$$= x - \frac{1}{6} x^3 + \frac{3}{40} x^5 + \dots$$