

L07b: Electromagnetic Potentials

EE3109 Applied Electromagnetics

Lecture Notes

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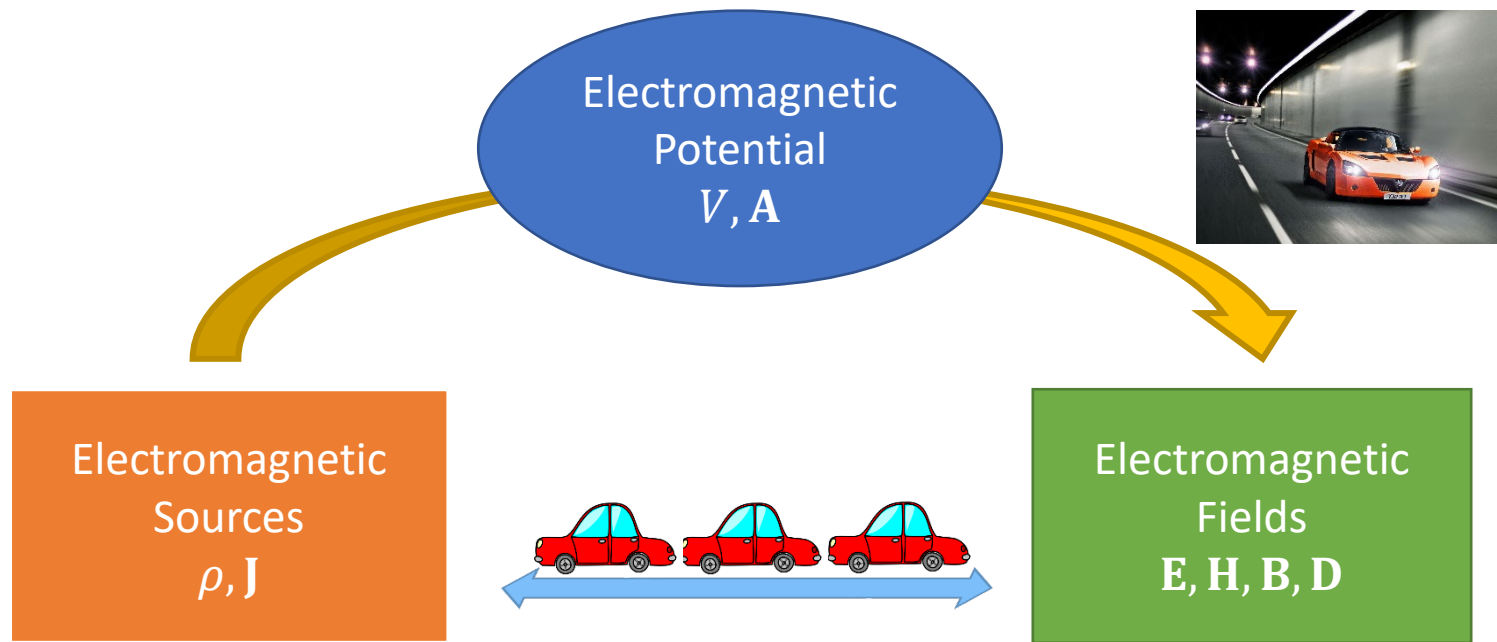
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Overview

- In this note we explain the motivation and demonstrate the application of electromagnetic potentials
- We cover the following major aspects
 - Defining electromagnetic (EM) potentials
 - Wave equations for EM potentials
 - Solving for EM fields with EM potentials
 - Example: Deriving the Biot-Savart Law from Maxwell's Equations

Electromagnetic Potential - Concept

- Electromagnetic potentials lead to more efficient calculations of the effect of sources on electromagnetic fields



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Electromagnetic Potentials – Definition

Consider $\nabla \cdot \mathbf{B} = 0$

Recalling the identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, we may define a Magnetic Vector Potential $\mathbf{A}$ such that

$$\nabla \times \mathbf{A} = \mathbf{B}$$

Substitute into the Faraday's law:

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial (\nabla \times \mathbf{A})}{\partial t} = -\nabla \times \frac{\partial \mathbf{A}}{\partial t} \\ \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) &= 0\end{aligned}$$

Now recall the identity $\nabla \times (\nabla V) = 0$, we may define a scalar potential V such that

$$-\nabla V = \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}$$

Note that, in the static limit, this definition becomes $-\nabla V = \mathbf{E}$, which is consistent with the electrostatic case)

Rearranging, we have

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

Now we apply the electromagnetic potentials to Ampere's Law:
(next page...)

$$\frac{1}{\mu} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \mu \mathbf{J} + \mu \varepsilon \frac{\partial}{\partial t} \left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t} \right)$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \nabla \left(\mu \varepsilon \frac{\partial V}{\partial t} \right) - \mu \varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

$$\nabla^2 \mathbf{A} - \mu \varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla \left(\nabla \cdot \mathbf{A} + \mu \varepsilon \frac{\partial V}{\partial t} \right)$$

We only defined $\nabla \times \mathbf{A} = \mathbf{B}$. We are at liberty to define $\nabla \cdot \mathbf{A}$ without affecting the value of the EM fields. *We can define $\nabla \cdot \mathbf{A}$ in a way that simplifies our problem!*

Lorentz Gauge: we define $\nabla \cdot \mathbf{A} = -\mu \varepsilon \frac{\partial V}{\partial t}$

The above equation becomes

$$\nabla^2 \mathbf{A} - \mu \varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}$$

(Non-homogeneous wave equation for $\mathbf{A}$)

Now look at Gauss' Law:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon} \\ -\nabla \cdot \left(\nabla V + \frac{\partial \mathbf{A}}{\partial t} \right) &= \frac{\rho}{\epsilon} \\ \nabla^2 V + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) &= -\frac{\rho}{\epsilon}\end{aligned}$$

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$$

(Non-homogeneous wave equation for V)

Remarks:

1. The two wave equations connect sources $(\rho, \mathbf{J})$ with potentials $(V, \mathbf{A})$
2. The waves have a speed of $v_p = 1/\sqrt{\mu\epsilon}$

Solving the differential equations:

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}$$

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$$

Phasor forms: ($k = \omega\sqrt{\mu\epsilon}$)

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$

$$\nabla^2 V + k^2 V = -\frac{\rho}{\epsilon}$$

(Separable) ↓

$$\begin{aligned}\nabla^2 A_x + k^2 A_x &= -\mu J_x \\ \nabla^2 A_y + k^2 A_y &= -\mu J_y \\ \nabla^2 A_z + k^2 A_z &= -\mu J_z\end{aligned}$$

All are non-homogeneous wave equations and should have the same solution form.

Solution to the Spherical Wave Equation:

* You are not responsible for material with this gray tab on the slide 

$$\nabla^2 V + k^2 V = -\frac{\rho}{\epsilon}$$

Since the equation is “linear and space-invariant”, we apply a Green’s function (impulse response) method. First assume the source ρ is a delta function (point charge) at location $\mathbf{r}'$ and find the answer, then construct the answer for a general $\rho(\mathbf{r})$ as an integral of individual delta-function sources.

Solution to the Spherical Wave Equation with delta source:

$$\nabla^2 V + k^2 V = -\frac{\delta^3(\mathbf{R})}{\epsilon} \quad (\text{defined } \mathbf{R} = \mathbf{r} - \mathbf{r}')$$

Solutions should be of the form $V = C \frac{e^{jkR}}{R}$, where $R = |\mathbf{R}|$, and C is a constant.

Verifying the Solution to the Spherical Wave Equation:

1. Calculate $\nabla^2 V + k^2 V$ for $\mathbf{R} \neq 0$:

$$\begin{aligned}\nabla^2 V + k^2 V &= \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) + k^2 V \\&= \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial}{\partial R} \left(C \frac{e^{-jkR}}{R} \right) \right) + k^2 C \frac{e^{-jkR}}{R} \\&= \frac{C}{R^2} \frac{\partial}{\partial R} \left(R^2 \left(-jk \frac{e^{-jkR}}{R} - \frac{e^{-jkR}}{R^2} \right) \right) + k^2 C \frac{e^{-jkR}}{R} \\&= -\frac{C}{R^2} \frac{\partial}{\partial R} \left(e^{-jkR} (jkR + 1) \right) + k^2 C \frac{e^{-jkR}}{R} \\&= -\frac{C}{R^2} \left(-jke^{-jkR} (jkR + 1) + jke^{-jkR} \right) + k^2 C \frac{e^{-jkR}}{R} \\&= -\frac{C}{R} \left(k^2 e^{-jkR} \right) + k^2 C \frac{e^{-jkR}}{R} = 0\end{aligned}$$

(As required)

Verifying the Solution to the Spherical Wave Equation (Cont'd):

2. We cannot evaluate either side of the function at $R = 0$, but we can take its **volumetric integral** over a **sphere of radius a , centered at $R = 0$** , and then take the **limit as $a \rightarrow 0$** .

Here we use v (lower case) to represent the volume

$$\begin{aligned} \int_{sph.} k^2 V dv &= \int_{sph.} Ck^2 \frac{e^{-jkR}}{R} dv = Ck^2 \int_0^a \frac{e^{-jkR}}{R} (4\pi R^2) dR \quad \text{(integration along } \theta \text{ and } \phi \text{ leads to this)} \\ &= 4\pi Ck^2 \int_0^a R e^{-jkR} dR = 4\pi Ck^2 \left\{ \left[\frac{jR}{k} e^{-jkR} \right]_0^a - \int_0^a \frac{j}{k} e^{-jkR} dR \right\} \quad \text{(integration by parts)} \\ &= j4\pi Cka e^{-jka} + 4\pi Ck^2 \left[\frac{1}{k^2} e^{-jka} \right]_0^a = j4\pi Cka e^{-jka} + 4\pi C(e^{-jka} - 1) \end{aligned}$$

Here both terms approach zero as $a \rightarrow 0$.

Verifying the Solution to the Spherical Wave Equation (Cont'd):

We define $\mathbf{P} = -\nabla V$, which means $\nabla \cdot \mathbf{P} = -\nabla \cdot \nabla V = -\nabla^2 V$

$$\text{Now } \mathbf{P} = -\nabla V = -\nabla \left(C \frac{e^{-jkR}}{R} \right) = C \frac{\partial}{\partial R} \left(\frac{e^{-jkR}}{R} \right) \hat{\mathbf{R}} = C \frac{e^{-jkR}}{R} \left(\frac{1}{R} + jk \right) \hat{\mathbf{R}}$$

$$\int_{sph.} \nabla^2 V \, dv = - \int_{sph.} \nabla \cdot \mathbf{P} \, dv = - \oint_{surf} \mathbf{P} \cdot d\mathbf{s} \quad \text{(Divergence theorem)}$$

Since V is radially symmetric and decays with $\mathbf{R}$, $\mathbf{P}$ always points in the direction of $\hat{\mathbf{R}}$.

$$\begin{aligned} \dots &= - \oint_{surf} \mathbf{P} \cdot d\mathbf{s} = 4\pi a^2 |\mathbf{P}(R = a)| = 4\pi a^2 \left[C \frac{e^{-jka}}{a} \left(\frac{1}{a} + jk \right) \right] \\ &= 4\pi C a e^{-jka} \left(\frac{1}{a} + jk \right) = 4\pi C e^{-jka} + j4\pi C k a e^{-jka} \end{aligned}$$

So $\lim_{a \rightarrow 0} \int_{sph.} \nabla^2 V \, dv = 4\pi C$ due to contribution from the first term.

Verifying the Solution to the Spherical Wave Equation (Cont'd):

Combining the final results from Slides 10 – 12, we see that on the LHS,

$$\int_{sph.} \nabla^2 V + k^2 V \, dv = 4\pi C$$

Now for the RHS: $\int_{sph.} \frac{\delta^3(\mathbf{R})}{\epsilon_0} \, dv = 1/\epsilon_0$ (property of the delta function).

Equating both sides, we see that $4\pi C = \frac{1}{\epsilon_0} \Rightarrow C = \frac{1}{4\pi\epsilon_0}$, i.e.

$$\nabla^2 V + k^2 V = -\frac{\delta^3(\mathbf{R})}{\epsilon_0} \Rightarrow V = \frac{1}{4\pi\epsilon_0} \left(\frac{e^{-jkR}}{R} \right)$$

Similar mathematics apply for the A equations:

$$\nabla^2 A_n + k^2 A_n = -\mu_0 \delta^3(\mathbf{R}) \Rightarrow A_n = \frac{\mu_0}{4\pi} \left(\frac{e^{-jkR}}{R} \right)$$

Where $n = x, y$ or z .

Solution to the Spherical Wave Equation:

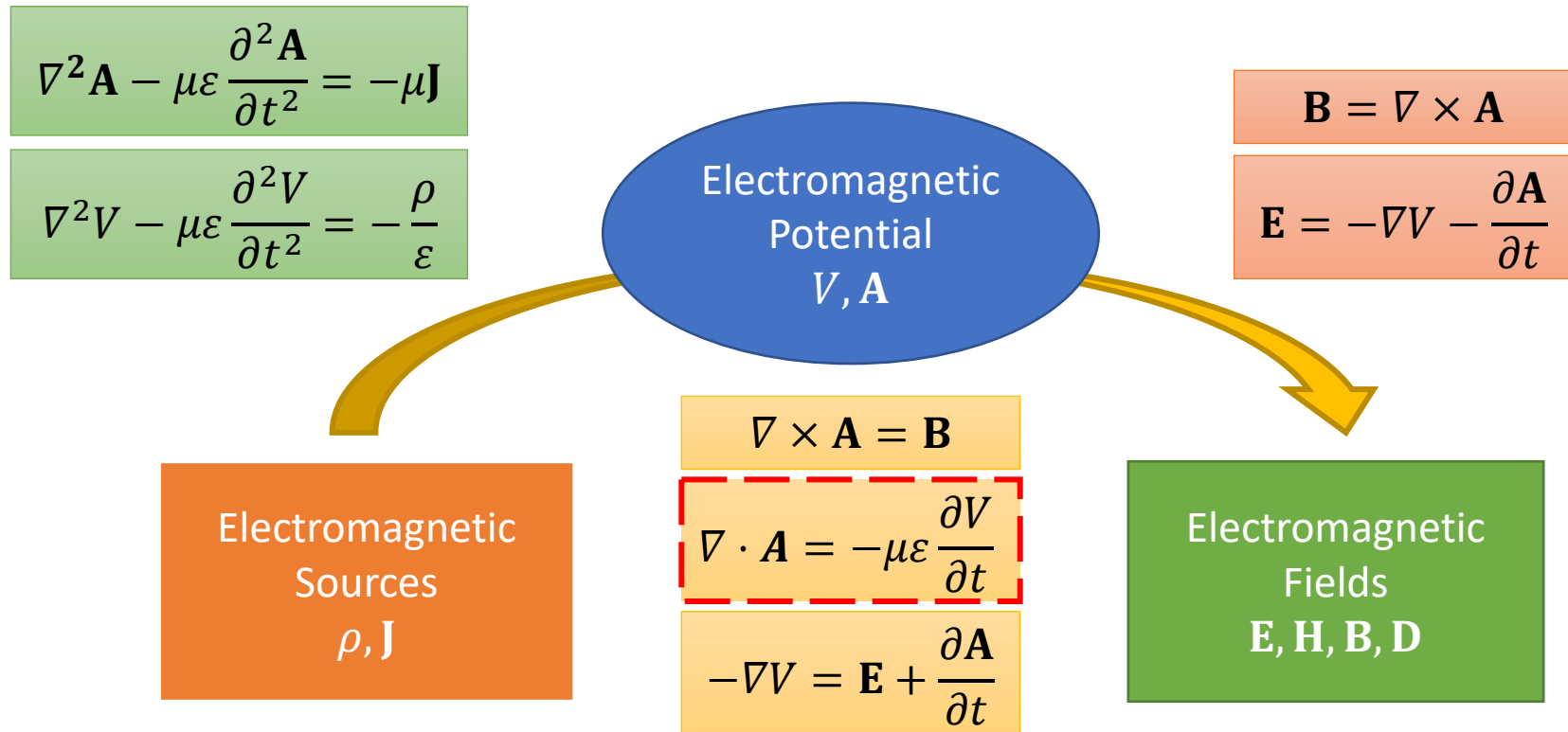
3. Now that we have obtained the “Green’s functions” to both differential equations, we can write the general answers to the differential equations by convolving the Green’s functions with the current/charge profile:

$$\begin{aligned}\nabla^2 V + k^2 V &= \frac{\rho(\mathbf{r})}{\varepsilon} \Rightarrow V(\mathbf{r}) = \frac{1}{4\pi\varepsilon} \int_{v'} \frac{e^{-jkR}}{R} \rho(\mathbf{r}') d\mathbf{r}' \\ \nabla^2 \mathbf{A} + k^2 \mathbf{A} &= -\mu \mathbf{J}(\mathbf{r}) \Rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_{v'} \frac{e^{-jkR}}{R} \mathbf{J}(\mathbf{r}') d\mathbf{r}'\end{aligned}$$

Where again, $R = |\mathbf{r} - \mathbf{r}'|$, and $\mathbf{r}'$ and v' denote the (current/voltage) source regions.

$\mathbf{A}(\mathbf{r})$ and $V(\mathbf{r})$ are called the **retarded magnetic and electric potentials** respectively.

Electromagnetic Potential - Summary



Obtaining EM Potentials from Sources

1. Potential Equations in Time Domain

$$\nabla^2 \mathbf{A}(\mathbf{r}, t) - \mu\epsilon \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} = -\mu \mathbf{J}(\mathbf{r})$$

$$\nabla^2 V(\mathbf{r}, t) - \mu\epsilon \frac{\partial^2 V(\mathbf{r}, t)}{\partial t^2} = -\frac{\rho(\mathbf{r})}{\epsilon}$$

2. Corresponding Phasors

$$\nabla^2 \mathbf{A}(\mathbf{r}) + k^2 \mathbf{A}(\mathbf{r}) = -\mu \mathbf{J}(\mathbf{r})$$

$$\nabla^2 V(\mathbf{r}) + k^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon}$$

$$(k = \omega\sqrt{\mu\epsilon})$$

3. Solutions to Differential Equations

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}(\mathbf{r}') e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dV'$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(\mathbf{r}') e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dV'$$

4. Back to Electromagnetic Field (Phasors)

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla V - j\omega \mathbf{A}$$

Example 1

6.29 A Hertzian dipole is a short conducting wire carrying an approximately constant current over its length l . If such a dipole is placed along the z -axis with its midpoint at the origin, and if the current flowing through it is $i(t) = I_0 \cos \omega t$, find

(a) The retarded vector potential $\mathbf{A}$

(b) Find the corresponding $\mathbf{B}$ field

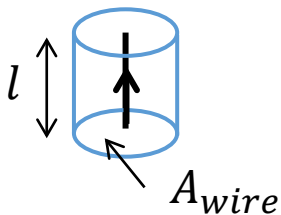
Solution:

First find the current density phasor: $I(\mathbf{r}) = I_0$

Then, write $\mathbf{J}(\mathbf{r})$: In this question the current density concentrates near the origin, so $\mathbf{J}(\mathbf{r}) \cong C_0 \delta^3(\mathbf{r}) \hat{\mathbf{z}}$.

$$\Rightarrow \mathbf{A}(\mathbf{r}) \cong \frac{\mu}{4\pi} \int_{V'} \frac{C_0 \delta^3(\mathbf{r}') e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \hat{\mathbf{z}} dV' = \frac{\mu C_0 e^{-jk|\mathbf{r}|}}{4\pi |\mathbf{r}|} \hat{\mathbf{z}}$$

But what is the value of C_0 ? It represents the value obtained by integrating the current density over the (infinitesimally) small volume where it is non-zero (i.e. the conducting wire): $C_0 = J \times A_{\text{wire}} \times l = I_0 l$. Substitution into the above gives:



$$\Rightarrow \mathbf{A}(\mathbf{r}) \cong \frac{\mu I_0 l}{4\pi r} e^{-jk \cdot r} \hat{\mathbf{z}} \quad (\text{where } r = |\mathbf{r}|)$$

Example 1 Cont'd

(b) Find the corresponding **B** field

Solution:

First find the current density phasor: $I(\mathbf{r}) = I_0$

Then, write $\mathbf{J}(\mathbf{r})$: In this question the current density concentrates near the origin, so $\mathbf{J}(\mathbf{r}) \cong C_0 \delta^3(\mathbf{r}) \hat{\mathbf{z}}$.

$$\Rightarrow \mathbf{A}(\mathbf{r}) \cong \frac{\mu}{4\pi} \int_{V'} \frac{C_0 \delta^3(\mathbf{r}') e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \hat{\mathbf{z}} dV' = \frac{\mu C_0}{4\pi} \frac{e^{-jk|\mathbf{r}|}}{|\mathbf{r}|} \hat{\mathbf{z}}$$

But what is the value of C_0 ? It represents the value obtained by integrating the current density over the (infinitesimally) small volume where it is non-zero (i.e. the conducting wire): $C_0 = J \times A_{\text{wire}} \times l = I_0 l$. Substitution into the above gives:

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) \cong \nabla \times \frac{\mu I_0 l}{4\pi r} e^{-jkr} \hat{\mathbf{z}} = \frac{\partial}{\partial y} \left(\frac{\mu I_0 l}{4\pi r} e^{-jkr} \right) \hat{\mathbf{x}} - \frac{\partial}{\partial x} \left(\frac{\mu I_0 l}{4\pi r} e^{-jkr} \right) \hat{\mathbf{y}}$$

Notice that $\mathbf{B}(\mathbf{r})$ points orthogonal to $\mathbf{A}(\mathbf{r})$ and $\mathbf{r}$. For a general case:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu}{4\pi} \int_{C'} \frac{I(\mathbf{r}') d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} e^{-jkR}$$

(Time-varying form of Biot-Savart Law)

Summary

- In this note we
 - Defined the EM potentials (Lorentz Gauge)
 - Derived wave equations for EM potentials
 - Found EM fields from known sources using EM potentials
 - Derived the Biot-Savart Law from Maxwell's Equations