

MA3001 Solution to Take-home Assignment 1

$$1. (a) \quad \frac{dy}{dx} = \frac{x^2 + 5y^2}{xy}, \quad y(1) = 2$$

$$= \frac{1 + 5\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}$$

Let $v = \frac{y}{x}$. Then $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$.

The DE becomes

$$v + x \frac{dv}{dx} = \frac{1 + 5v^2}{v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 + 5v^2}{v} - v$$

$$= \frac{1 + 5v^2 - v^2}{v}$$

$$= \frac{1 + 4v^2}{v}$$

$$\Rightarrow \int \frac{v}{1 + 4v^2} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \frac{1}{8} \int \frac{8v}{1 + 4v^2} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \frac{1}{8} \ln(1 + 4v^2) = \ln|x| + C$$

$$\Rightarrow \frac{1}{8} \ln\left[1 + 4\left(\frac{y}{x}\right)^2\right] = \ln|x| + C$$

With $y(1) = 2$, $\frac{1}{8} \ln\left[1 + 4\left(\frac{2}{1}\right)^2\right] = \ln 1 + C$

$$\Rightarrow C = \frac{1}{8} \ln 17$$

$\therefore$ The particular solution is

$$\frac{1}{8} \ln\left[1 + 4\left(\frac{y}{x}\right)^2\right] = \ln|x| + \frac{1}{8} \ln 17$$

i.e. $\ln\left[1 + 4\left(\frac{y}{x}\right)^2\right] = \ln(x^8) + \ln 17$

i.e. $1 + 4\left(\frac{y}{x}\right)^2 = 17x^8$

$$1(b) \quad \frac{dy}{dx} = \frac{x+2y-3}{3x+6y+2} = \frac{(x+2y)-3}{3(x+2y)+2}$$

$$\text{Let } v = x+2y. \text{ Then } \frac{dv}{dx} = 1 + 2 \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(\frac{dv}{dx} - 1 \right)$$

The DE becomes

$$\frac{1}{2} \left(\frac{dv}{dx} - 1 \right) = \frac{v-3}{3v+2}$$

$$\Rightarrow \frac{dv}{dx} - 1 = \frac{2v-6}{3v+2}$$

$$\begin{aligned} \Rightarrow \frac{dv}{dx} &= \frac{2v-6}{3v+2} + 1 \\ &= \frac{2v-6+(3v+2)}{3v+2} \\ &= \frac{5v-4}{3v+2} \end{aligned}$$

$$\Rightarrow \int \frac{3v+2}{5v-4} dv = \int dx$$

$$\Rightarrow \int \frac{\frac{3}{5}(5v-4) + \frac{12}{5} + 2}{5v-4} dv = \int dx$$

$$\Rightarrow \int \left(\frac{3}{5} + \frac{22}{5} \cdot \frac{1}{5v-4} \right) dv = \int dx$$

$$\Rightarrow \frac{3}{5}v + \frac{22}{25} \ln|5v-4| = x + C$$

$\therefore$ The general solution is

$$\frac{3}{5}(x+2y) + \frac{22}{25} \ln|5(x+2y)-4| = x + C$$

$$1(c) \quad \frac{dy}{dx} + \frac{y}{x} = \left(\frac{x^4}{e^{2x}} \right) y^3$$

Multiply both sides by y^{-3} :

$$y^{-3} \frac{dy}{dx} + \frac{1}{x} y^{-2} = \frac{x^4}{e^{2x}}$$

$$\text{Let } v = y^{-2}. \text{ Then } \frac{dv}{dx} = -2y^{-3} \frac{dy}{dx} \Rightarrow y^{-3} \frac{dy}{dx} = -\frac{1}{2} \frac{dv}{dx}.$$

$\therefore$ The DE becomes

$$-\frac{1}{2} \frac{dv}{dx} + \frac{1}{x} v = \frac{x^4}{e^{2x}}$$

$$\Rightarrow \frac{dv}{dx} - \frac{2}{x} v = \frac{-2x^4}{e^{2x}}$$

$$\text{Integrating factor : } I(x) = e^{\int -\frac{2}{x} dx} = e^{-2 \ln x} = e^{\ln(x^{-2})} = x^{-2}$$

$$\begin{aligned} \therefore \frac{d}{dx}(x^{-2}v) &= \frac{-2x^4}{e^{2x}} \cdot x^{-2} \\ &= -2x^2 e^{-2x} \end{aligned}$$

$$\begin{aligned} \Rightarrow x^{-2}v &= \int -2x^2 e^{-2x} dx \\ &= \int x^2 d(e^{-2x}) \\ &= x^2 e^{-2x} - \int e^{-2x} d(x^2) \\ &= x^2 e^{-2x} - 2 \int x e^{-2x} dx \\ &= x^2 e^{-2x} + \int x d(e^{-2x}) \\ &= x^2 e^{-2x} + x e^{-2x} - \int e^{-2x} dx \\ &= x^2 e^{-2x} + x e^{-2x} + \frac{1}{2} e^{-2x} + C \end{aligned}$$

$\therefore$ The general solution is

$$\frac{1}{x^2 y^2} = x^2 e^{-2x} + x e^{-2x} + \frac{1}{2} e^{-2x} + C$$

$$1(d) \quad (ye^{2xy} + x + 1) + (xe^{2xy} - 3y^2) \frac{dy}{dx} = 0$$

$$\text{Let } M(x,y) = ye^{2xy} + x + 1 \quad \text{and} \quad N(x,y) = xe^{2xy} - 3y^2.$$

$$\frac{\partial M}{\partial y} = e^{2xy} + 2xye^{2xy} = \frac{\partial N}{\partial x}$$

$\therefore$ The DE is exact.

Then there exists $u(x,y)$ s.t.

$$\begin{cases} \frac{\partial u}{\partial x} = ye^{2xy} + x + 1 & \text{--- ①} \\ \frac{\partial u}{\partial y} = xe^{2xy} - 3y^2 & \text{--- ②} \end{cases}$$

Integrating ① w.r.t. x :

$$u(x,y) = \frac{1}{2} e^{2xy} + \frac{1}{2} x^2 + x + g(y)$$

Partially differentiate w.r.t. y :

$$\frac{\partial u}{\partial y} = xe^{2xy} + g'(y)$$

$$\text{Comparing with ②, } g'(y) = -3y^2$$

$$\Rightarrow g(y) = -y^3$$

$\therefore$ The general solution is

$$\frac{1}{2} e^{2xy} + \frac{1}{2} x^2 + x - y^3 = C$$

2. (a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0$, $y(0) = 2$ & $y'(0) = 1$.

Auxiliary equation : $\alpha^2 + 2\alpha + 10 = 0$
 $\Rightarrow (\alpha + 1)^2 - 1^2 + 10 = 0$
 $\Rightarrow (\alpha + 1)^2 = -9$
 $\Rightarrow \alpha = -1 \pm 3i$

$\therefore$ The general solution is

$$y = e^{-x}(A \cos 3x + B \sin 3x)$$

where A, B are arbitrary constants.

$$\begin{aligned} y' &= -e^{-x}(A \cos 3x + B \sin 3x) + e^{-x}(-3A \sin 3x + 3B \cos 3x) \\ &= e^{-x}[(-A + 3B) \cos 3x + (-3A - B) \sin 3x] \end{aligned}$$

With $y(0) = 2$ & $y'(0) = 1$,

$$\begin{cases} A = 2 \\ -A + 3B = 1 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = 1 \end{cases}$$

$\therefore$ The particular solution is

$$y = e^{-x}(2 \cos 3x + \sin 3x)$$

$$2(b) \quad y^{(5)} + 8y^{(3)} + 16y' = 0$$

$$\text{Auxiliary equation : } \alpha^5 + 8\alpha^3 + 16\alpha = 0$$

$$\Rightarrow \alpha(\alpha^4 + 8\alpha^2 + 16) = 0$$

$$\Rightarrow \alpha(\alpha^2 + 4)^2 = 0$$

$$\Rightarrow \alpha = 0 \quad \text{or} \quad \alpha^2 = -4, -4$$

$$\Rightarrow \alpha = 0, \pm 2i, \pm 2i$$

$\therefore$ The general solution is

$$y = A + (Bx + C) \cos 2x + (Dx + E) \sin 2x,$$

where A, B, C, D, E are arbitrary constants.

$$(c) \quad y'' - 4y' + 4y = (x+1)e^{2x} + 13 \cos 3x$$

$$\text{Auxiliary equation : } \alpha^2 - 4\alpha + 4 = 0$$

$$\Rightarrow (\alpha - 2)^2 = 0$$

$$\Rightarrow \alpha = 2, 2$$

$\therefore$ The complementary function is

$$y_c = (Ax + B)e^{2x}$$

where A, B are arbitrary constants.

$$\text{Let } y_p = x^2(ax + b)e^{2x} + C_1 \cos 3x + C_2 \sin 3x$$

$$= (ax^3 + bx^2)e^{2x} + C_1 \cos 3x + C_2 \sin 3x$$

$$\text{Then } y_p' = [2ax^3 + (3a+2b)x^2 + 2bx] e^{2x} - 3C_1 \sin 3x + 3C_2 \cos 3x$$

$$\text{and } y_p'' = [4ax^3 + (12a+4b)x^2 + (6a+8b)x + 2b] e^{2x} - 9C_1 \cos 3x - 9C_2 \sin 3x$$

Substitute these into the DE :

$$\begin{aligned} & [4ax^3 + (12a+4b)x^2 + (6a+8b)x + 2b] e^{2x} - 9C_1 \cos 3x - 9C_2 \sin 3x \\ & - 4 \{ [2ax^3 + (3a+2b)x^2 + 2bx] e^{2x} - 3C_1 \sin 3x + 3C_2 \cos 3x \} \\ & + 4 [(ax^3 + bx^2) e^{2x} + C_1 \cos 3x + C_2 \sin 3x] \\ & = (x+1) e^{2x} + 13 \cos 3x \end{aligned}$$

$$\begin{aligned} \Rightarrow (6ax + 2b) e^{2x} + (-5C_1 - 12C_2) \cos 3x + (12C_1 - 5C_2) \sin 3x \\ = (x+1) e^{2x} + 13 \cos 3x \end{aligned}$$

Comparing coefficients :

$$\begin{cases} 6a = 1 \\ 2b = 1 \\ -5C_1 - 12C_2 = 13 \\ 12C_1 - 5C_2 = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{6} \\ b = \frac{1}{2} \\ -60C_1 - 144C_2 = 156 \\ 60C_1 - 25C_2 = 0 \end{cases}$$

$$\Rightarrow a = \frac{1}{6}, b = \frac{1}{2}, C_1 = -\frac{5}{13}, C_2 = -\frac{12}{13}$$

$$\therefore y_p = x^2 \left(\frac{1}{6}x + \frac{1}{2} \right) e^{2x} - \frac{5}{13} \cos 3x - \frac{12}{13} \sin 3x$$

$\therefore$ The general solution is

$$y = (Ax + B) e^{2x} + x^2 \left(\frac{1}{6}x + \frac{1}{2} \right) e^{2x} - \frac{5}{13} \cos 3x - \frac{12}{13} \sin 3x$$