
EE 3121
Differential Equations for Electrical Engineering

Lecture 6: Higher-Order Linear Ordinary Differential Equations



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Summary

This chapter has considered linear ODEs of order greater than two. The general form is

$$\frac{d^n y}{dt^n} + a_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n(t) y = g(t),$$

where $n > 2$.

- For linear time invariant ODEs of order higher than two, when a solution r to the characteristic equation has multiplicity m , there are m fundamental solutions $y_k(t)$ to the ODE with the form $t^k e^{rt}$, where $0 \leq k \leq m - 1$. This includes the case where r is complex.

Summary

For a linear ODE of order n , the kernel is given by

$$K(t, \tau) = \sum_{i=1}^n y_i(t) W_i(\tau) / W(\tau),$$

where $y_i(t)$ are the ODEs n fundamental solutions, the Wronskian

$$W(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \\ \dot{y}_1 & \dot{y}_2 & \cdots & \dot{y}_{n-1} & \dot{y}_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_{n-1}^{(n-1)} & y_n^{(n-1)} \end{vmatrix}. \quad (1)$$

Summary

$y_j^{(k)}(\tau)$ is the k th derivative of the j th fundamental solution, and $W_i(\tau)$ is the Wronskian with i th column replaced with

$$\begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}$$

Summary

- For linear time-invariant ODEs, it remains true for higher order ODEs, as it did for first and second order ones, that if one or more of the coefficients in the ODE is negative then the system is unstable.
- All the coefficients in a higher-order ODE may be positive and yet the system be unstable.
- Solutions to a higherorder ODE are stable and damped if all the roots of its characteristic equation have negative real parts.

Solved Problems

SP6.1 Solve:

$$\frac{d^3y}{dt^3} + 8\frac{d^2y}{dt^2} + 45\frac{dy}{dt} + 116y = 0 \quad (2)$$

$$y(0) = 2 \quad \dot{y}(0) = -1 \quad \ddot{y}(0) = 3$$

Solution:

Since this is a linear time-invariant equation, we try $y(t) = e^{rt}$, resulting in the characteristic equation

$$r^3 + 8r^2 + 45r + 116 = 0. \quad (3)$$

Solved Problems

- Equation (3) must have at least one real root. The reasons are as below:
 - The roots of a polynomial with real coefficients must be either real or occur in complex conjugate pairs (Recall [Section 1.1.2.2](#)).
 - The only possibilities for the solutions of Equation (3) are (a) three real roots or (b) one real root and one complex conjugate pair.

Call

$$P_3(r) = r^3 + 8r^2 + 45r + 116.$$

We evaluate $P_3(r)$ for various values of r .

Solved Problems

- The easiest value for which to evaluate $P_3(r)$ is $r = 0$. $P_0(r) = 116$.
- Values of r greater than zero would only increase $P_3(r)$, so we must examine $r < 0$.
- Trying $r = -1$ yields $P_3(-1) = 78$, which is in the right direction (i.e., better than 116) but still far from zero.
- Trying $r = -10$ and find $P_3(-10) = -534$, a change of sign.
- $P_3(r)$ is a continuous function, somewhere between $r = -1$ and $r = -10$ lies at least one root. (There could be three.)

Solved Problems

- Trying $r = -5$ yields $P_3(-5) = -34$, so there is at least one real root between $r = -1$ and $r = -5$.
- Trying $r = -4$ yields $P_3(-4) = 0$ and we have found a root; that is, $r = -4$ is a solution of $P_3(r) = r^3 + 8r^2 + 45r + 116 = 0$
- We can employ the *NewtonRaphson* method when, as is the usual case, the roots are decimal numbers, not integers, and the root must be accurate to at least several decimal places.

Solved Problems

Once the first root r_1 is obtained, we divide through, forming a polynomial

$$P_2(r) = \frac{P_3(r)}{r - r_1}.$$

We do this by long division:

$$\begin{array}{r} r^2 + 4r + 29 \\ r + 4 \overline{) r^3 + 8r^2 + 45r + 116} \\ \underline{r^3 + 4r^2} \\ 4r^2 + 45r \\ \underline{4r^2 + 16r} \\ 29r + 116 \end{array}$$

Solved Problems

Hence $P_2(r) = r^2 + 4r + 29$. Completing the square, we see that

$$P_2(r) = r^2 + 4r + 29 = (r + 2)^2 + 5^2,$$

and so

$$P_3(r) = r^3 + 8r^2 + 45r + 116 = (r + 4)((r + 2)^2 + 5^2),$$

and the roots of Equation (3) are $r_1 = -4$, $r_2 = -2 + i5$, and $r_3 = -2 - i5$.

Hence the fundamental solutions to Equation (2) are $y_1(t) = e^{-4t}$, $y_2(t) = e^{-2t} \cos 5t$, and $y_3(t) = e^{-2t} \sin 5t$ and the general solution to Equation (2) is

Solved Problems

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) = c_1 e^{-4t} + c_2 e^{-2t} \cos 5t + c_3 e^{-2t} \sin 5t. \quad (4)$$

Now

$$\frac{dy}{dt} = -4c_1 e^{-4t} - 2c_2 e^{-2t} \cos 5t - 5c_2 e^{-2t} \sin 5t - 2c_3 e^{-2t} \sin 5t + 5c_3 e^{-2t} \cos 5t, \quad (5)$$

and

$$\begin{aligned} \frac{d^2 y}{dt^2} = & 16c_1 e^{-4t} + 4c_2 e^{-2t} \cos 5t + 20c_2 e^{-2t} \sin 5t - 25c_2 e^{-2t} \cos 5t \\ & + 4c_3 e^{-2t} \sin 5t - 20c_3 e^{-2t} \cos 5t - 25c_3 e^{-2t} \sin 5t \end{aligned} \quad (6)$$

Solved Problems

From the initial conditions and Equations (4), (5), and (6),

$$y(0) = 2 = c_1 + c_2$$

$$\frac{dy}{dt}(0) = -1 = -4c_1 - 2c_2 + 5c_3 \quad (7)$$

$$\frac{d^2y}{dt^2}(0) = 3 = 16c_1 - 21c_2 - 20c_3.$$

This is three linear equations in three unknowns. The solutions are

$$c_1 = 57/29$$

$$c_2 = 1/29$$

$$c_3 = 201/145.$$

Solved Problems

Hence the solution to Equation (2) is

$$y(t) = \left(\frac{57}{29}\right) e^{-4t} + \left(\frac{1}{29}\right) e^{-2t} \cos 5t + \left(\frac{201}{145}\right) e^{-2t} \sin 5t. \quad (8)$$

Solved Problems

SP6.2 Find the kernel $K(t - \tau)$ for the system governed by

$$\frac{d^3y}{dt^3} + 16\frac{d^2y}{dt^2} + 81\frac{dy}{dt} + 126y = g(t) \quad (9)$$

Solution:

- Section 6.3.1 shows that the kernel derives from the systems fundamental solutions.
- Since Equation (9) is a linear time-invariant equation we find the fundamental solutions by trying the solution $y(t) = e^{rt}$ in the homogeneous equation

Solved Problems

$$\frac{d^3y}{dt^3} + 16\frac{d^2y}{dt^2} + 81\frac{dy}{dt} + 126y = 0. \quad (10)$$

We quickly arrive at the characteristic equation

$$r^3 + 16r^2 + 81r + 126 = 0. \quad (11)$$

- We could again employ the method used in Solved Problem SP6.1 to factor this polynomial.

According to the web-based 'root-calculator', the solutions of Equation (11) are $r_1 = -3$, $r_2 = -6$, and $r_3 = -7$.

How do we check this?

Solved Problems

Multiplying the factors out:

$$\begin{aligned}(r + 3)(r + 6)(r + 7) &= (r + 3)(r^2 + 13r + 42) \\ &= r^3 + 13r^2 + 42r + 3r^2 + 39r + 126 \\ &= r^3 + 16r^2 + 81r + 126.\end{aligned}$$

Comparing this result to Equation (11) we see that the web-based root calculator has given us the correct result. Hence the fundamental solutions of Equation (10) are $y_1(t) = e^{-3t}$, $y_2(t) = e^{-6t}$ and $y_3(t) = e^{-7t}$. Now,

$$K(t, \tau) = \sum_{i=1}^3 y_i(t) W_i(\tau) / W(\tau), \quad (12)$$

Solved Problems

From

$$r^n + a_1 r^{n-1} + a_2 r^{n-2} + \dots + a_n = 0.$$

$$\begin{aligned} W(\tau) &= \begin{vmatrix} y_1(\tau) & y_2(\tau) & y_3(\tau) \\ \dot{y}_1(\tau) & \dot{y}_2(\tau) & \dot{y}_3(\tau) \\ \ddot{y}_1(\tau) & \ddot{y}_2(\tau) & \ddot{y}_3(\tau) \end{vmatrix} = \begin{vmatrix} e^{-3\tau} & e^{-6\tau} & e^{-7\tau} \\ -3e^{-3\tau} & -6e^{-6\tau} & -7e^{-7\tau} \\ 9e^{-3\tau} & 36e^{-6\tau} & 49e^{-7\tau} \end{vmatrix} \\ &= e^{-3\tau} \begin{vmatrix} -6e^{-6\tau} & -7e^{-7\tau} \\ 36e^{-6\tau} & 49e^{-7\tau} \end{vmatrix} - e^{-6\tau} \begin{vmatrix} -3e^{-3\tau} & -7e^{-7\tau} \\ 9e^{-3\tau} & 49e^{-7\tau} \end{vmatrix} \\ &\quad + e^{-7\tau} \begin{vmatrix} -3e^{-3\tau} & -6e^{-6\tau} \\ 9e^{-3\tau} & 36e^{-6\tau} \end{vmatrix} \\ &= e^{-3\tau} [(-6e^{-6\tau})(49e^{-7\tau}) - (36e^{-6\tau})(-7e^{-7\tau})] \\ &\quad - e^{-6\tau} [(-3e^{-3\tau})(49e^{-7\tau}) - (9e^{-3\tau})(-7e^{-7\tau})] \\ &\quad + e^{-7\tau} [(-3e^{-3\tau})(36e^{-6\tau}) - (9e^{-3\tau})(-6e^{-6\tau})] \\ &= e^{-3\tau} e^{-6\tau} e^{-7\tau} ([-294 + 252] - [-147 + 63] + [-108 + 51]). \end{aligned}$$

Solved Problems

$$W(\tau) = -12e^{-16\tau}. \quad (13)$$

Similarly,

$$\begin{aligned} W_1(\tau) &= \begin{vmatrix} 0 & y_2(\tau) & y_3(\tau) \\ 0 & \dot{y}_2(\tau) & \dot{y}_3(\tau) \\ 1 & \ddot{y}_2(\tau) & \ddot{y}_3(\tau) \end{vmatrix} = \begin{vmatrix} 0 & e^{-6\tau} & e^{-7\tau} \\ 0 & -6e^{-6\tau} & -7e^{-7\tau} \\ 1 & 36e^{-6\tau} & 49e^{-7\tau} \end{vmatrix} \\ &= e^{-6\tau}(-7e^{-7\tau}) - (-6e^{-6\tau})e^{-7\tau}, \end{aligned}$$

$$W_1(\tau) = -e^{-13\tau}. \quad (14)$$

Solved Problems

$$\begin{aligned} W_2(\tau) &= \begin{vmatrix} y_1(\tau) & 0 & y_3(\tau) \\ \dot{y}_1(\tau) & 0 & \dot{y}_3(\tau) \\ \ddot{y}_1(\tau) & 1 & \ddot{y}_3(\tau) \end{vmatrix} = \begin{vmatrix} e^{-3\tau} & 0 & e^{-7\tau} \\ -3e^{-3\tau} & 0 & -7e^{-7\tau} \\ 9e^{-3\tau} & 1 & 49e^{-7\tau} \end{vmatrix} \\ &= (e^{-3\tau}(-7e^{-7\tau}) - (-3e^{-3\tau})e^{-7\tau}), \end{aligned}$$

$$W_2(\tau) = 4e^{-10\tau}. \quad (15)$$

$$\begin{aligned} W_3(\tau) &= \begin{vmatrix} y_1(\tau) & y_2(\tau) & 0 \\ \dot{y}_1(\tau) & \dot{y}_2(\tau) & 0 \\ \ddot{y}_1(\tau) & \ddot{y}_2(\tau) & 0 \end{vmatrix} = \begin{vmatrix} e^{-3\tau} & e^{-6\tau} & 0 \\ -3e^{-3\tau} & -6e^{-6\tau} & 0 \\ 9e^{-3\tau} & 36e^{-6\tau} & 1 \end{vmatrix} \\ &= e^{-3\tau}(-6e^{-6\tau}) - (-3e^{-3\tau})e^{-6\tau}, \end{aligned}$$

$$W_3(\tau) = -3e^{-9\tau}. \quad (16)$$

Solved Problems

Substituting Equations (13) through (16) into Equation (12) we arrive at

$$K(t, \tau) = (e^{-3t}(-e^{-13\tau}) + e^{-6t}(4e^{-10\tau}) + e^{-7t}(-3e^{-9\tau})) / (-12e^{-16\tau})$$

$$K(t, \tau) = (e^{-3t}(-e^{3\tau}) + e^{-6t}(4e^{6\tau}) + e^{-7t}(-3e^{7\tau})) / (-12)$$

$$K(t, \tau) = (1/12) \left(e^{-3(t-\tau)} - 4e^{-6(t-\tau)} + 3e^{-7(t-\tau)} \right). \quad (17)$$

- As a rudimentary check on Equation (17), since Equation (9) is a linear time-invariant ODE, its kernel should be of the form $K(t, \tau) = K(t - \tau)$.

Solved Problems

SP6.3 Find the steady-state solution to

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 2e^{2t} \sin 3t \quad (18)$$

Solution:

From [Table 5.1](#), we try

$$y(t) = e^{2t}(c \sin 3t + d \cos 3t). \quad (19)$$

Differentiating Equation (19) once, twice, and three times:

$$\frac{dy}{dt} = e^{2t}(2c \sin 3t + 2d \cos 3t) + e^{2t}(3c \cos 3t - 3d \sin 3t). \quad (20)$$

Solved Problems

$$\frac{d^2y}{dt^2} = e^{2t}(-5c \sin 3t - 5d \cos 3t) + e^{2t}(12c \cos 3t - 12d \sin 3t). \quad (21)$$

$$\frac{d^3y}{dt^3} = e^{2t}(-46c \sin 3t - 46d \cos 3t) + e^{2t}(9c \cos 3t - 9d \sin 3t). \quad (22)$$

Substituting Equations (20) through (22) into Equation (19) and combining coefficients of $\sin 3t$ and $\cos 3t$ we have

$$e^{2t}(-43c - 45d) \sin 3t + e^{2t}(45c + 43d) \cos 3t = 2e^{2t} \sin 3t.$$

Solved Problems

We can divide both sides through by e^{rt} , since it is never zero, and then we are left with

$$(-43c - 45d) \sin 3t + (45c + 43d) \cos 3t = 2 \sin 3t. \quad (23)$$

For Equation (23) to be true for all t implies that

$$\begin{aligned} -43c - 45d &= 2, \\ 45c + 43d &= 0. \end{aligned} \quad (24)$$

Equation (24) are two linear equations in two unknowns. The solutions are

$$\begin{aligned} c &= 43/88, \\ d &= -45/88. \end{aligned}$$

Solved Problems

So the steady-state solution to Equation (18) is

$$y(t) = e^{2t} \left((43/88) \sin 3t - (45/88) \cos 3t \right). \quad (25)$$

END