

MA1201 Semester A 2022

Suggested Solutions to Assignment 2

1

$$\begin{aligned} \frac{d}{dx} \left(\int_{x^3}^{e^{2x}} \frac{\sin(3y)}{\sqrt{y^2 + \cos(3y) + 1}} dy \right) &= (e^{2x})' \frac{\sin(3e^{2x})}{\sqrt{e^{4x} + \cos(3e^{2x}) + 1}} - (x^3)' \frac{\sin(3x^3)}{\sqrt{x^6 + \cos(3x^3) + 1}} \\ &= \frac{2e^{2x} \sin(3e^{2x})}{\sqrt{e^{4x} + \cos(3e^{2x}) + 1}} - \frac{3x^2 \sin(3x^3)}{\sqrt{x^6 + \cos(3x^3) + 1}} \end{aligned}$$

2a Let $t = \sqrt{x}$. Then

$$\begin{aligned} \int \frac{\cot^{-1}(\sqrt{x})}{\sqrt{x}(1+x)} dx &= \int \frac{\cot^{-1}(t)}{t(1+t^2)} 2t dt = 2 \int \frac{\cot^{-1}(t)}{1+t^2} dt = -2 \int \cot^{-1}(t) d \cot^{-1}(t) \\ &= -(\cot^{-1}(t))^2 + C = -(\cot^{-1}(\sqrt{x}))^2 + C. \end{aligned}$$

2b Let $t = e^x$. Then

$$\int_{\ln 2}^{\ln 3} \frac{1}{e^x + e^{-x}} dx = \int_2^3 \frac{1}{t + t^{-1}} \frac{dt}{t} = \int_2^3 \frac{1}{t^2 + 1} dt = \tan^{-1}(t) \Big|_2^3 = \tan^{-1}(3) - \tan^{-1}(2) = \tan^{-1}\left(\frac{1}{7}\right).$$

2c Denote $I_{2n} = \int x^{2n} \sin(5x) dx$. And $I_0 = \int \sin(5x) dx = -\frac{1}{5} \cos(5x)$.

For any integer $n \geq 1$, integration by parts yields

$$\begin{aligned} I_{2n} &= -\frac{1}{5} x^{2n} \cos(5x) + \frac{1}{5} \int 2nx^{2n-1} \cos(5x) dx = -\frac{1}{5} x^{2n} \cos(5x) + \frac{2n}{25} x^{2n-1} \sin(5x) - \frac{2n}{25} \int (2n-1)x^{2n-2} \sin(5x) dx \\ &= -\frac{1}{5} x^{2n} \cos(5x) + \frac{1}{25} 2nx^{2n-1} \sin(5x) - \frac{1}{25} 2n(2n-1) I_{2n-2} \\ &= (2n)! \left\{ \sum_{k=0}^n (-1)^{k+1} \frac{x^{2n-2k}}{(2n-2k)! 5^{2k+1}} \cos(5x) + \sum_{k=0}^{n-1} (-1)^k \frac{x^{2n-2k-1}}{(2n-2k-1)! 5^{2k+2}} \sin(5x) \right\} + C. \end{aligned}$$

Thus when $n = 4$, we have

$$\begin{aligned} I_8 &= -\frac{1}{5} x^8 \cos(5x) + \frac{8}{25} x^7 \sin(5x) + \frac{56}{125} x^6 \cos(5x) - \frac{336}{625} x^5 \sin(5x) - \frac{336}{625} x^4 \cos(5x) + \frac{1344}{3125} x^3 \sin(5x) \\ &\quad + \frac{4032}{15625} x^2 \cos(5x) - \frac{8064}{78125} x \sin(5x) - \frac{8064}{390625} \cos(5x) + C. \end{aligned}$$

2d First, by polynomial division we obtain

$$x^5 = (9x^3 + 21x^2 + 25x + 13) \underbrace{\left(\frac{1}{9}x^2 - \frac{7}{27}x + \frac{8}{27} \right)}_{\text{Quotient}} + \underbrace{\left(-\frac{32}{27}x^2 - \frac{109}{27}x - \frac{104}{27} \right)}_{\text{Remainder}}.$$

Next partial fraction decomposition is available.

$$\frac{32x^2 + 109x + 104}{(x+1)(9x^2 + 12x + 13)} = \frac{A}{x+1} + \frac{Bx+C}{9x^2 + 12x + 13} \Rightarrow 32x^2 + 109x + 104 = A(9x^2 + 12x + 13) + (Bx+C)(x+1)$$

After using method of undetermined coefficients, we get $A = \frac{27}{10}, B = \frac{77}{10}, C = \frac{689}{10}$. Therefore,

$$\begin{aligned} \int \frac{5x^5}{(x+1)(9x^2 + 12x + 13)} dx &= 5 \int \left(\frac{1}{9}x^2 - \frac{7}{27}x + \frac{8}{27} - \frac{1}{27} \frac{32x^2 + 109x + 104}{(x+1)(9x^2 + 12x + 13)} \right) dx \\ &= 5 \int \left(\frac{1}{9}x^2 - \frac{7}{27}x + \frac{8}{27} - \frac{1}{10(x+1)} - \frac{77x+689}{270(9x^2 + 12x + 13)} \right) dx \\ &= \frac{5}{27}x^3 - \frac{35}{54}x^2 + \frac{40}{27}x - \frac{1}{2} \ln|x+1| - \frac{77}{972} \int \frac{d(9x^2 + 12x + 13)}{9x^2 + 12x + 13} - \frac{1913}{162} \int \frac{1}{9(x + \frac{2}{3})^2 + 1} dx \\ &= \frac{5}{27}x^3 - \frac{35}{54}x^2 + \frac{40}{27}x - \frac{1}{2} \ln|x+1| - \frac{77}{972} \ln(9x^2 + 12x + 13) - \frac{1913}{1458} \tan^{-1}(x + \frac{2}{3}) + C. \end{aligned}$$

3

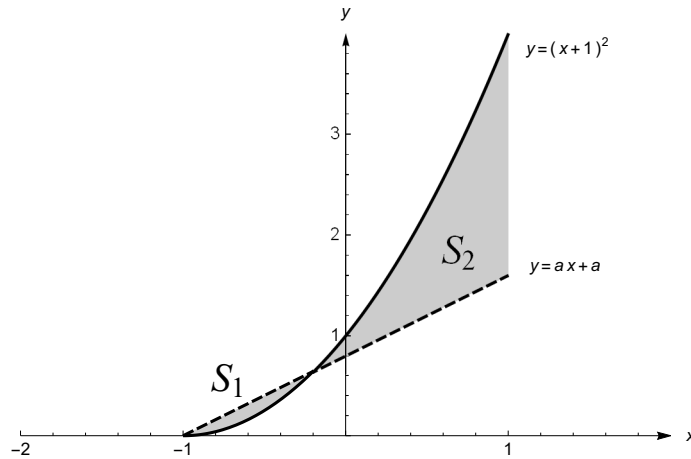
$$V_a = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^2 dx = \frac{\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos(2x)) dx = \frac{\pi}{2} \left(x - \frac{1}{2} \sin(2x) \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi^2}{2}$$

$$V_b = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\pi - x) \cos x dx = 2\pi^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx = 4\pi^2$$

$$V_c = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} \right)^2 - \left(\frac{3}{2} - \cos x \right)^2 dx = 3\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx - \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^2 dx = 6\pi - \frac{\pi^2}{2}$$

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The area S_1, S_2 are shown in the following figure.



One can find the intersection points of the curve and the line with solving simultaneous equations.

$$\begin{cases} y = (x+1)^2 \\ y = ax + a \end{cases} \Rightarrow \begin{cases} x = -1 \\ y = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = a-1 \\ y = a^2 \end{cases}$$

Hence,

$$S(a) := S_1 + S_2 = \int_{-1}^{a-1} ((ax+a) - (x+1)^2) dx + \int_{a-1}^1 ((x+1)^2 - (ax+a)) dx = \frac{a^3}{3} - 2a + \frac{8}{3}.$$

Since $S'(a) = a^2 - 2 < 0$ for any $0 \leq a \leq 1$, $S(a)$ attains the minimum value at $a = 1$, i.e., $S(1) = 1$.

When revolving around the horizontal line $y = -1$, the volume reads

$$V = \pi \int_{-1}^0 (x+1 - (-1))^2 - ((x+1)^2 - (-1))^2 dx + \pi \int_0^1 ((x+1)^2 - (-1))^2 - (x+1 - (-1))^2 dx = 6\pi.$$

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$$\begin{aligned} S &= \int_0^{2\pi} 2\pi(3 - (2 - \cos t)) \sqrt{(t + \sin t)'^2 + (2 - \cos t)'^2} dt = \int_0^{2\pi} 2\pi(1 + \cos t) \sqrt{2 + 2\cos t} dt \\ &= 2\sqrt{2}\pi \int_0^{2\pi} (1 + \cos t)^{\frac{3}{2}} dt = 2\sqrt{2}\pi \int_0^{2\pi} (1 + 2\cos^2 \frac{t}{2} - 1)^{\frac{3}{2}} dt = 8\pi \int_0^{2\pi} \left| \cos^3 \frac{t}{2} \right| dt = 16\pi \int_0^{\pi} |\cos^3 t| dt \\ &= 32\pi \int_0^{\pi/2} \cos^3 t dt = 32\pi \int_0^{\pi/2} (1 - \sin^2 t) \cos t dt = 32\pi \int_0^{\pi/2} \cos t dt - 32\pi \int_0^{\pi/2} \sin^2 t d\sin t \\ &= 32\pi \left[\sin t - \frac{1}{3} \sin^3 t \right]_0^{\pi/2} = \frac{64}{3}\pi \end{aligned}$$

