

MA 1201 Semester B 2018/19

Assignment 3 — Due at 5 pm, 18/4/2019 (Thursday) in tutorial classes

Instructions:

- Please show your work. Unsupported answers will receive **NO** credits.
- Make sure you write down the correct lecture session (A/B/C/D/E/F/G/H) you have registered for, together with your full name and student ID on the front page of your answer script. Hand your solution to your TA in tutorial class before the deadline.
- In case your tutorial class is scheduled on days other than Thursday, you may choose to hand your solution to your TA in one of the tutorial classes scheduled on Thursday.
- **NO** late homework will be accepted. Homework submitted to wrong tutorial sessions will **NOT** be graded and will receive **0 POINTS**.

1. (15 points) Find the area of the surface generated by revolving the curve $y = \sqrt{2x - x^2}$, $1/2 \leq x \leq 3/2$, about the x -axis.

Solution. The abstract surface area formula states that (see page 203 of the supplementary notes)

$$S = \int 2\pi\rho \, ds,$$

where the radius of revolution ρ and the arc length differential ds are given by (see page 189–191 of the supplementary notes for the definition of the arc length differential)

$$\rho = y \text{ (rotate about } x\text{-axis)}, \quad ds = \sqrt{dx^2 + dy^2}.$$

Since the curve is described by a function of x , the surface area should be expressed as an integral of x , which in turn requires that ρ and ds be expressed in terms of x , as:

$$\begin{aligned} \rho &= y = \sqrt{2x - x^2}, \\ ds &= \sqrt{dx^2 + \left(\frac{dy}{dx} dx\right)^2} = \sqrt{1 + [y'(x)]^2} dx, \end{aligned}$$

where

$$\begin{aligned} 1 + [y'(x)]^2 &= 1 + \left(\frac{1}{2}(2x - x^2)^{-1/2} \cdot (2 - 2x)\right)^2 = 1 + \left(\frac{1 - x}{\sqrt{2x - x^2}}\right)^2 = 1 + \frac{(1 - x)^2}{2x - x^2} \\ &= \frac{1}{2x - x^2} \cdot [(2x - x^2) + (1 - x)^2] = \frac{1}{2x - x^2} \cdot [(2x - x^2) + (1 - 2x + x^2)] = \frac{1}{2x - x^2}. \end{aligned}$$

It then follows that

$$S = \int_{1/2}^{3/2} 2\pi\sqrt{2x - x^2} \cdot \left(\frac{1}{\sqrt{2x - x^2}} dx\right) = 2\pi \int_{1/2}^{3/2} 1 \, dx = 2\pi x \Big|_{1/2}^{3/2} = 2\pi.$$

2. (10 points) Suppose a complex number z satisfies the equation

$$\frac{1+z}{1-z} = e^{i\theta},$$

for some $\theta \in (\pi, 2\pi)$. Find the complex number z and express the result in Euler's form with principal argument.

Solution. Rearranging the equation and solving for z , one finds (note that $e^{i\theta} \neq -1$ since $\theta \neq \pi$)

$$1+z = e^{i\theta}(1-z) \implies (e^{i\theta}+1)z = e^{i\theta}-1 \implies z = \frac{e^{i\theta}-1}{e^{i\theta}+1}.$$

To express z in Euler's form, write $1 = e^{i0}$ and factor out $e^{i[(\theta+0)/2]} = e^{i(\theta/2)}$ from both the numerator and the denominator to obtain

$$z = \frac{e^{i\theta} - e^{i0}}{e^{i\theta} + e^{i0}} = \frac{e^{i(\theta/2)}[e^{i(\theta/2)} - e^{-i(\theta/2)}]}{e^{i(\theta/2)}[e^{i(\theta/2)} + e^{-i(\theta/2)}]} = \frac{e^{i(\theta/2)} - e^{-i(\theta/2)}}{e^{i(\theta/2)} + e^{-i(\theta/2)}} = \frac{2i \sin(\theta/2)}{2 \cos(\theta/2)} = i \tan \frac{\theta}{2}.$$

Since $\theta \in (\pi, 2\pi)$, the angle $(\theta/2)$ lies in quadrant II, which implies that

$$\tan \frac{\theta}{2} < 0 \quad \text{and hence} \quad \tan \frac{\theta}{2} = -\left| \tan \frac{\theta}{2} \right|.$$

It then follows that

$$\begin{aligned} z &= i \tan \frac{\theta}{2} = i \cdot (-1) \cdot \left| \tan \frac{\theta}{2} \right| = e^{i(\pi/2)} \cdot e^{i\pi} \cdot \left| \tan \frac{\theta}{2} \right| \\ &= \left| \tan \frac{\theta}{2} \right| e^{i(3\pi/2)} = \left| \tan \frac{\theta}{2} \right| e^{i(3\pi/2-2\pi)} = \left| \tan \frac{\theta}{2} \right| e^{-i(\pi/2)}, \end{aligned}$$

where an additional 2π was subtracted out from the argument because $(3\pi/2)$ is not a principal angle (lying between $-\pi$ and π), while $(3\pi/2 - 2\pi)$ is.

3. (10 points) Suppose a complex number z satisfies the equation

$$2z^2 - (3+8i)z - (m+4i) = 0.$$

Given that m is a real constant and that one of the two solutions is real, find the two solutions of the above complex quadratic equation. (Hint: The real solution is easy to find. To find the other solution, use the fact that the two solutions z_1, z_2 of the quadratic equation $az^2 + bz + c = 0$ satisfy $z_1 + z_2 = -b/a$.)

Solution. Let $z_1 = \alpha$ be the real solution of the equation and $z_2 = \beta + \gamma i$ be the other solution, which is possibly complex. The substitution of $z = z_1 = \alpha$ into the equation yields (recall α and m are both real numbers)

$$2\alpha^2 - (3+8i)\alpha - (m+4i) = 0 \implies (2\alpha^2 - 3\alpha - m) - i(8\alpha + 4) = 0 \implies \begin{cases} 2\alpha^2 - 3\alpha - m = 0 \\ 8\alpha + 4 = 0 \end{cases},$$

which, in particular, shows that $\alpha = -1/2$ and $m = 2$. To find the other solution, recall that the two solutions z_1, z_2 of the quadratic equation $az^2 + bz + c = 0$ satisfy

$$z_{1,2} = \frac{1}{2a} (-b \pm \sqrt{b^2 - 4ac}), \quad \text{so that} \quad z_1 + z_2 = -\frac{b}{a}.$$

It then follows that

$$z_2 = -\frac{b}{a} - z_1 = -\frac{1}{2} [-(3+8i)] - \left(-\frac{1}{2}\right) = 2+4i.$$

4. (15 points) Find all the complex solutions of the equation

$$z^4 + (\sqrt{3} - 3i)(1+z)^4 = 0.$$

Express the results in Cartesian form (you may express your answer as a function of k , e.g. $z = e^{i(k\pi/2)} = \cos(k\pi/2) + i\sin(k\pi/2)$, $k = 0, 1, 2, 3$, without explicitly evaluating the expression for each k).

Solution. Dividing through the equation by $(1+z)^4$ and rearranging, one finds

$$\frac{z^4}{(1+z)^4} + (\sqrt{3} - 3i) = 0, \quad \text{or} \quad \left(\frac{z}{1+z}\right)^4 = -\sqrt{3} + 3i.$$

To proceed, express $(-\sqrt{3} + 3i)$ in Euler's form (note that the number lies in quadrant II):

$$-\sqrt{3} + 3i = 2\sqrt{3}e^{i(2\pi/3)},$$

and solve the equation for $z/(1+z)$. This yields (see page 218–223 of the supplementary notes for the general procedure for finding powers and roots of complex numbers)

$$\begin{aligned} \frac{z}{1+z} &= (-\sqrt{3} + 3i)^{1/4} = [2\sqrt{3}e^{i(2\pi/3)}]^{1/4} = [2\sqrt{3}e^{i(2\pi/3+2k\pi)}]^{1/4} \\ &= (2\sqrt{3})^{1/4}e^{i(2\pi/3+2k\pi)/4} = (2\sqrt{3})^{1/4}e^{i(\pi/6+k\pi/2)}, \quad k = 0, 1, 2, 3. \end{aligned}$$

If $w_k = re^{i\theta_k} = (2\sqrt{3})^{1/4}e^{i(\pi/6+k\pi/2)}$ denotes the right side of the above equation, then

$$\frac{z}{1+z} = w_k \implies z = w_k(1+z) \implies z = \frac{w_k}{1-w_k},$$

and thus (see page 228 of the supplementary notes for basic properties of complex conjugate)

$$\begin{aligned} z &= \frac{w_k}{1-w_k} = \frac{re^{i\theta_k}}{1-re^{i\theta_k}} = \frac{re^{i\theta_k}}{1-re^{i\theta_k}} \cdot \frac{(1-re^{-i\theta_k})}{(1-re^{-i\theta_k})} = \frac{re^{i\theta_k}(1-re^{-i\theta_k})}{|1-re^{i\theta_k}|^2} = \frac{re^{i\theta_k} - r^2}{|1-re^{i\theta_k}|^2} \\ &= \frac{r(\cos\theta_k + i\sin\theta_k) - r^2}{|1-r(\cos\theta_k + i\sin\theta_k)|^2} = \frac{(r\cos\theta_k - r^2) + ir\sin\theta_k}{(1-r\cos\theta_k)^2 + (r\sin\theta_k)^2} = \alpha_k + \beta_k i, \quad k = 0, 1, 2, 3, \end{aligned}$$

where

$$\begin{aligned} \alpha_k &= \frac{r\cos\theta_k - r^2}{(1-r\cos\theta_k)^2 + (r\sin\theta_k)^2} = \frac{(2\sqrt{3})^{1/4}\cos(\pi/6+k\pi/2) - (2\sqrt{3})^{1/2}}{[1 - (2\sqrt{3})^{1/4}\cos(\pi/6+k\pi/2)]^2 + [(2\sqrt{3})^{1/4}\sin(\pi/6+k\pi/2)]^2}, \\ \beta_k &= \frac{r\sin\theta_k}{(1-r\cos\theta_k)^2 + (r\sin\theta_k)^2} = \frac{(2\sqrt{3})^{1/4}\sin(\pi/6+k\pi/2)}{[1 - (2\sqrt{3})^{1/4}\cos(\pi/6+k\pi/2)]^2 + [(2\sqrt{3})^{1/4}\sin(\pi/6+k\pi/2)]^2}. \end{aligned}$$

5. (15 points) Let

$$A = \begin{pmatrix} 3 & 1 & -2 \\ -2 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

- (a) (5 points) Evaluate the determinant of A by the cofactor expansion and compute $\det(A^{-2})$.

Solution. To find $\det A$, expand the determinant along the third row which contains the least number of nonzero entries. This yields

$$\begin{aligned}\det A &= 0 \cdot (-1)^{3+1} \begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} + 1 \cdot (-1)^{3+2} \begin{vmatrix} 3 & -2 \\ -2 & 2 \end{vmatrix} + 1 \cdot (-1)^{3+3} \begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} \\ &= 0 + (-1) \cdot [3 \cdot 2 - (-2) \cdot (-2)] + 1 \cdot [3 \cdot 2 - (-2) \cdot 1] = 0 - 2 + 8 = 6.\end{aligned}$$

The properties of determinant then imply that

$$\det(A^{-2}) = (\det A)^{-2} = 6^{-2} = \frac{1}{36}.$$

In particular, there is no need to find A^2 and its inverse $A^{-2} = (A^2)^{-1}$ explicitly.

Remark. The “power law” $\det(A^{-2}) = (\det A)^{-2}$ is a consequence of the following simple observations. The product rule for determinants implies that, for any $n \times n$ matrix A and any positive integer m ,

$$\det(A^m) = \det(\underbrace{A \cdot A \cdots A}_{m \text{ copies}}) = \underbrace{\det A \cdot \det A \cdots \det A}_{m \text{ copies}} = (\det A)^m.$$

In addition, for any $n \times n$ invertible matrix A ,

$$1 = \det I_n = \det(A \cdot A^{-1}) = \det A \cdot \det(A^{-1}) \implies \det(A^{-1}) = \frac{1}{\det A} = (\det A)^{-1}.$$

Thus for any $n \times n$ invertible matrix A and any positive integer m ,

$$\det(A^{-m}) = \det[(A^m)^{-1}] = [\det(A^m)]^{-1} = [(\det A)^m]^{-1} = (\det A)^{-m}.$$

In particular, for $m = 2$,

$$\det(A^{-2}) = (\det A)^{-2}.$$

- (b) (10 points) Find all values of λ such that $\det(A - \lambda I_3) = 0$, where I_3 is the 3×3 identity matrix.

Solution. Note that

$$A - \lambda I_3 = \begin{pmatrix} 3 - \lambda & 1 & -2 \\ -2 & 2 - \lambda & 2 \\ 0 & 1 & 1 - \lambda \end{pmatrix},$$

so to find $\det(A - \lambda I_3)$, it suffices to expand the determinant along the third row, which contains the least number of nonzero entries. This yields

$$\begin{aligned}\det(A - \lambda I_3) &= 0 \cdot (-1)^{3+1} \begin{vmatrix} 1 & -2 \\ 2 - \lambda & 2 \end{vmatrix} + 1 \cdot (-1)^{3+2} \begin{vmatrix} 3 - \lambda & -2 \\ -2 & 2 \end{vmatrix} + (1 - \lambda) \cdot (-1)^{3+3} \begin{vmatrix} 3 - \lambda & 1 \\ -2 & 2 - \lambda \end{vmatrix} \\ &= 0 + (-1) [(3 - \lambda) \cdot 2 - (-2) \cdot (-2)] + (1 - \lambda) [(3 - \lambda) \cdot (2 - \lambda) - (-2) \cdot 1] \\ &= 0 - (2 - 2\lambda) + (1 - \lambda)(\lambda^2 - 5\lambda + 8) = (1 - \lambda)[(-2) + (\lambda^2 - 5\lambda + 8)] \\ &= (1 - \lambda)(\lambda^2 - 5\lambda + 6) = -(\lambda - 1)(\lambda - 2)(\lambda - 3).\end{aligned}$$

Thus the solutions of the equation $\det(A - \lambda I_3) = 0$ (i.e. the eigenvalues of the matrix A) are given by

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3.$$

6. (15 points) Consider the linear system

$$\begin{aligned}x_1 - 2x_2 + 3x_3 &= 1 \\2x_1 - x_2 + 2cx_3 &= 2 \\3x_1 - 3x_2 + c^2x_3 &= c\end{aligned}$$

Find all possible values of c such that the system

- (a) (5 points) has a unique solution.
- (b) (5 points) has infinitely many solutions.
- (c) (5 points) has no solutions.

Solution. Writing the system in matrix form $Ax = b$ and applying Gaussian elimination to the augmented matrix $[A|b]$, one finds

$$[A|b] = \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & -1 & 2c & 2 \\ 3 & -3 & c^2 & c \end{array} \right) \xrightarrow[R_2 - 2R_1]{R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 3 & 2c-6 & 0 \\ 0 & 3 & c^2-9 & c-3 \end{array} \right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 3 & 2c-6 & 0 \\ 0 & 0 & c^2-2c-3 & c-3 \end{array} \right).$$

In order for the system to have a unique solution, the reduced matrix must have the same number of pivots (equations) as columns (unknowns), which requires (see page 255–257 of the supplementary notes for a detailed discussion on the existence and uniqueness of solutions of a linear system, based on the results of Gaussian elimination)

$$c^2 - 2c - 3 = (c - 3)(c + 1) \neq 0, \quad \text{or} \quad c \neq -1, c \neq 3.$$

If this condition is violated, i.e. if $c = -1$ or $c = 3$, then the reduced matrix becomes

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 3 & 2c-6 & 0 \\ 0 & 0 & 0 & c-3 \end{array} \right),$$

and the system has a solution if and only if $c - 3 = 0$ as well (so that the last row is entirely zero). This implies $c = 3$, in which case the reduced matrix becomes

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

and the system has infinitely many solutions. Finally, if $c = -1$, the reduced matrix becomes

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 0 & -4 \end{array} \right),$$

which admits no solutions. In conclusion, the system

- (a) has a unique solution when $c \in \mathbb{R} \setminus \{-1, 3\}$;
 (b) has infinitely many solutions when $c = 3$; and
 (c) has no solutions when $c = -1$.

7. (20 points) Let

$$A = \begin{pmatrix} 0 & -1 & 5 & -3 \\ 1 & -2 & 3 & -4 \\ 2 & -3 & 1 & -5 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix}.$$

- (a) (15 points) Solve the linear system $Ax = b$ by Gaussian elimination and express the general solution in vector form.

Solution. Applying Gaussian elimination to the augmented matrix $[A|b]$, one finds (see page 252–254 of the supplementary notes for a detailed description of the Gaussian elimination procedure, including the choice of pivot columns and pivots)

$$\begin{aligned} [A|b] &= \left(\begin{array}{cccc|c} 0 & -1 & 5 & -3 & 8 \\ 1 & -2 & 3 & -4 & 5 \\ 2 & -3 & 1 & -5 & 2 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 5 \\ 0 & -1 & 5 & -3 & 8 \\ 2 & -3 & 1 & -5 & 2 \end{array} \right) \\ &\xrightarrow{R_3 - 2R_1} \left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 5 \\ 0 & -1 & 5 & -3 & 8 \\ 0 & 1 & -5 & 3 & -8 \end{array} \right) \xrightarrow{R_3 + R_2} \left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 5 \\ 0 & -1 & 5 & -3 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right). \end{aligned}$$

Since there is no rows of the form $(0 \ 0 \ 0 \ 0 | \alpha \neq 0)$ in the reduced matrix, the system $Ax = b$ is consistent, i.e. it has at least one solution. Since there are more unknowns (4) than equations (2) in the reduced system, the system is expected to have infinitely many solutions. To find all these solutions, write the reduced system as

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 4x_4 &= 5 \\ -x_2 + 5x_3 - 3x_4 &= 8 \end{aligned}$$

and identify the pivot variables x_1, x_2 together with the free variables x_3, x_4 (see page 261–264 of the supplementary notes for the choice of pivot and free variables). Solving for the pivot variables from the equations in terms of the free variables, one finds

$$\begin{aligned} x_2 &= -8 + 5x_3 - 3x_4, \\ x_1 &= 5 + 2x_2 - 3x_3 + 4x_4 = 5 + 2(-8 + 5x_3 - 3x_4) - 3x_3 + 4x_4 = -11 + 7x_3 - 2x_4. \end{aligned}$$

The general solution of the system $Ax = b$, expressed in vector form, is then given by

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -11 + 7x_3 - 2x_4 \\ -8 + 5x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -11 \\ -8 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ 5 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

- (b) (5 points) Write down the corresponding homogenous system $Ax = 0$ explicitly and determine all non-trivial solutions from (a) without resolving the system.

Solution. The homogenous system associated with the linear system $Ax = b$ is

$$\begin{aligned} -x_2 + 5x_3 - 3x_4 &= 0 \\ x_1 - 2x_2 + 3x_3 - 4x_4 &= 0 \\ 2x_1 - 3x_2 + x_3 - 5x_4 &= 0 \end{aligned}$$

Its general solution is given by

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 7 \\ 5 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ -3 \\ 0 \\ 1 \end{pmatrix},$$

which is that part of the solution of the nonhomogeneous system $Ax = b$ that contains free variables.

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