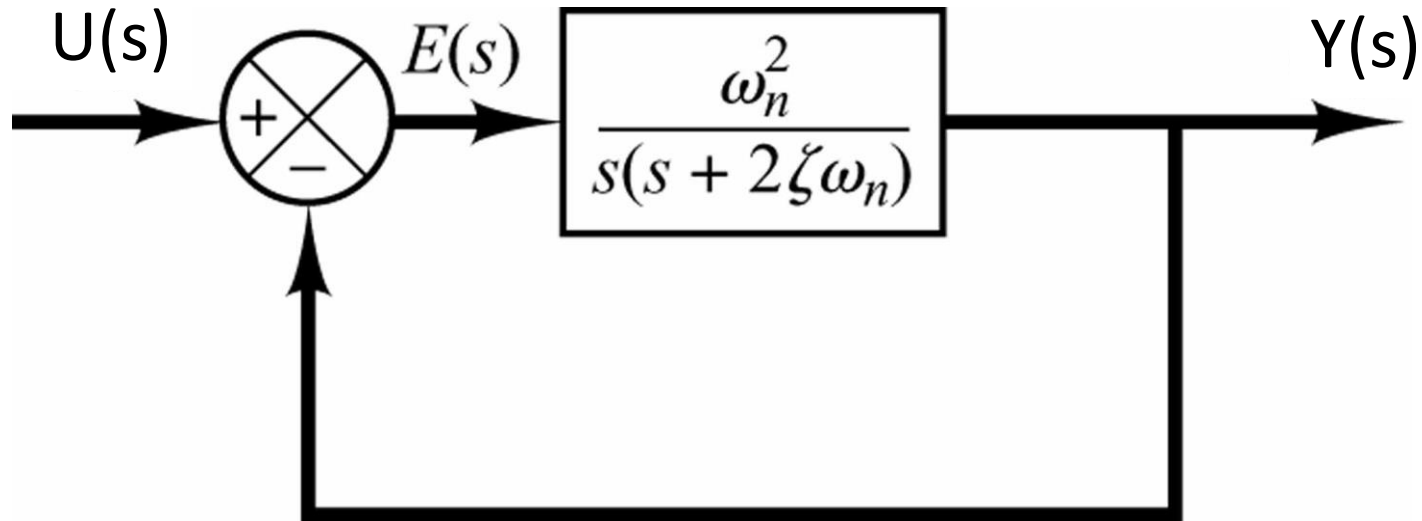


EE3114 Systems and Control

5. Root Locus Analysis and Design

Semester B, 2017-18

Recall: Standard form of second-order systems



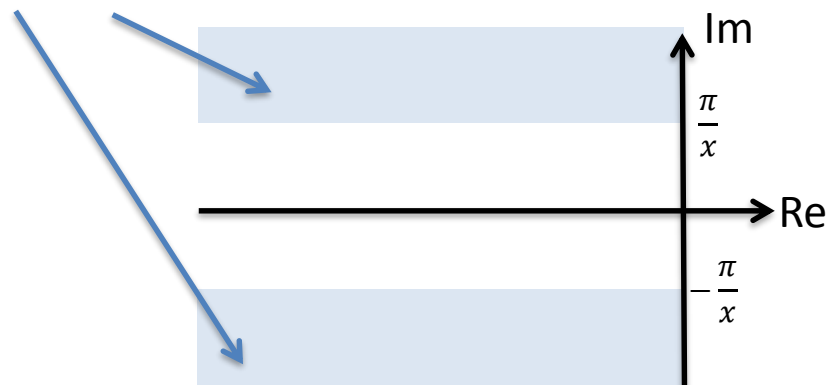
$$\frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Damping ratio

Un-damped natural frequency

Recall: Allowable Region on s-Plane

- $p = -\sigma \pm j\omega_d$ (Cartesian representation)
 $= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ ((ω_n, ζ) representation)
- Peak Time: $t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
 - Depending only on *imaginary* part of complex poles
 - Specification: $t_p \leq x \rightarrow \omega_n\sqrt{1-\zeta^2} \geq \frac{\pi}{x}$
 - Pole pair **Allowable Region** to satisfy $t_p \leq x$



- Percent Overshoot

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

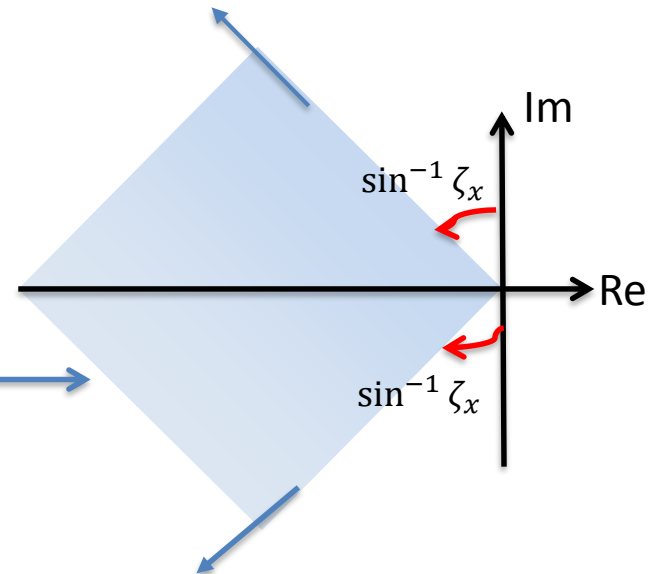
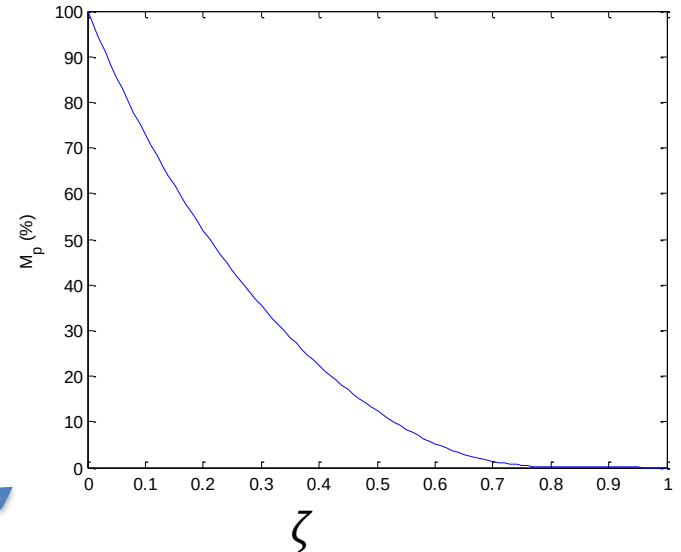
- Depending only on ζ
- Specification: $M_p < x\%$

→ $\zeta \geq \zeta_x \rightarrow \theta \geq \sin^{-1} \zeta_x$

- ζ_x obtained from figure
- Or, by solving equation

$$-\frac{\zeta_x\pi}{\sqrt{1-\zeta_x^2}} < \ln(x\%)$$

- Pole pair **Allowable Region** to satisfy $M_p < x\%$

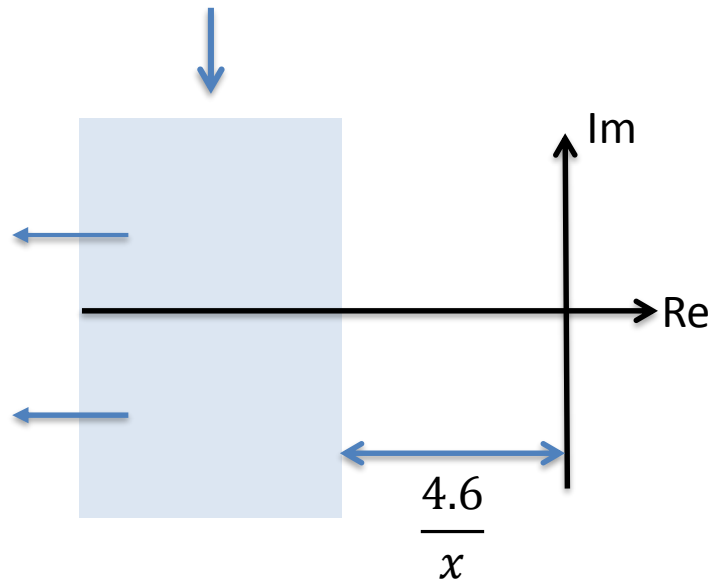


- 1% **Settling Time**:
$$t_s = \frac{4.6}{\zeta\omega_n} = \frac{4.6}{\sigma}$$

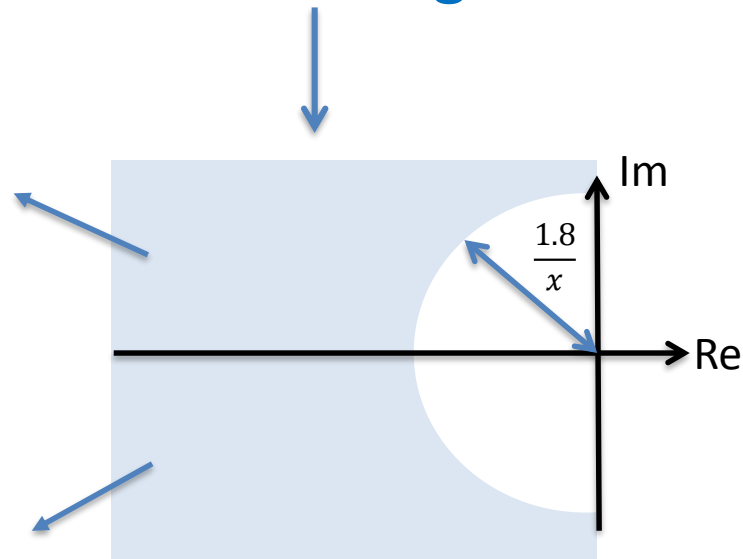
- Depending only on *real* part of complex poles

- Specification: $t_s \leq x \rightarrow \zeta\omega_n \geq \frac{4.6}{x}$

- Pole pair **Allowable Region** to satisfy $t_s \leq x$

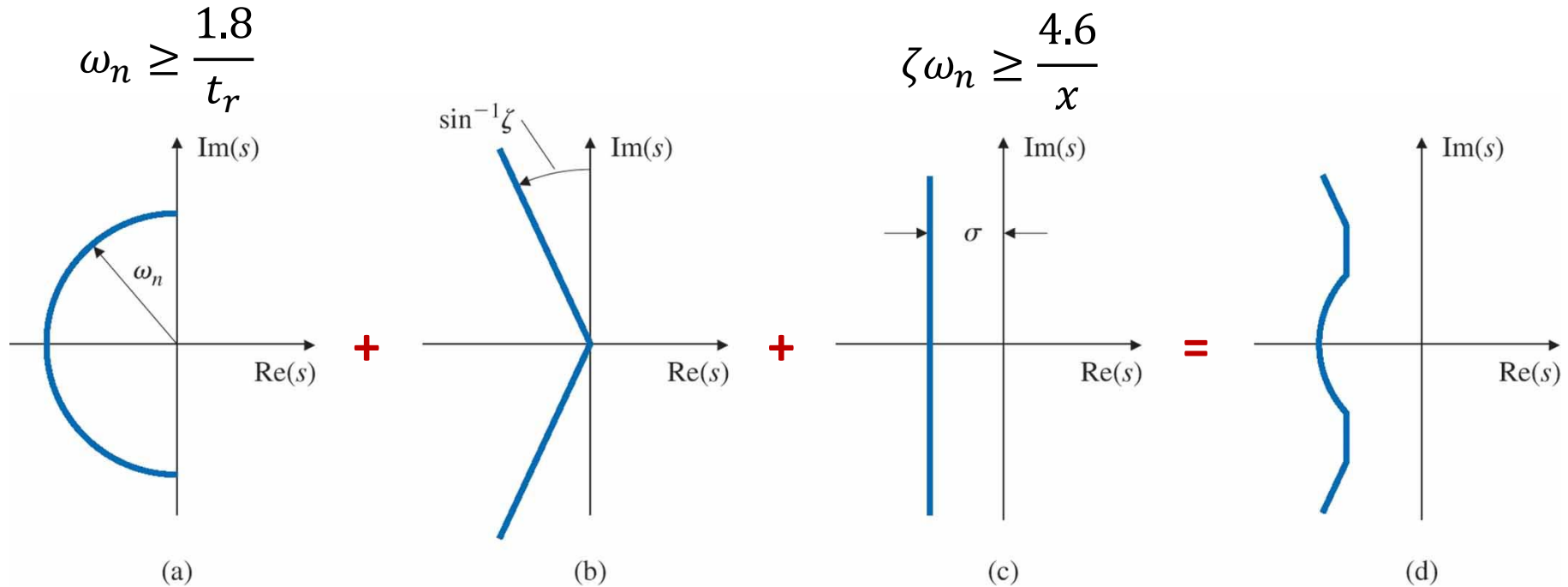


- Rise time $t_r \cong \frac{1.8}{\omega_n}$
 - Depending only on ω_n
 - Specification: $t_r \leq x \rightarrow \omega_n \geq \frac{1.8}{x}$
 - Pole pair **Allowable Region** to satisfy $t_r \leq x$



Allowable Region on s-Plane

(delineated all by transient requirements)



Rise Time

Overshoot

1% Settling Time

Composite of all
3 requirements

Example: Transformation of Specifications to the s-Plane

Find the Allowable Region in the s-plane for the poles of a Transfer Function of a system if its response requirements are:
 $t_r \leq 0.6 \text{ sec}$, $M_p \leq 10\%$ and 1% settling time $t_s \leq 3 \text{ sec}$

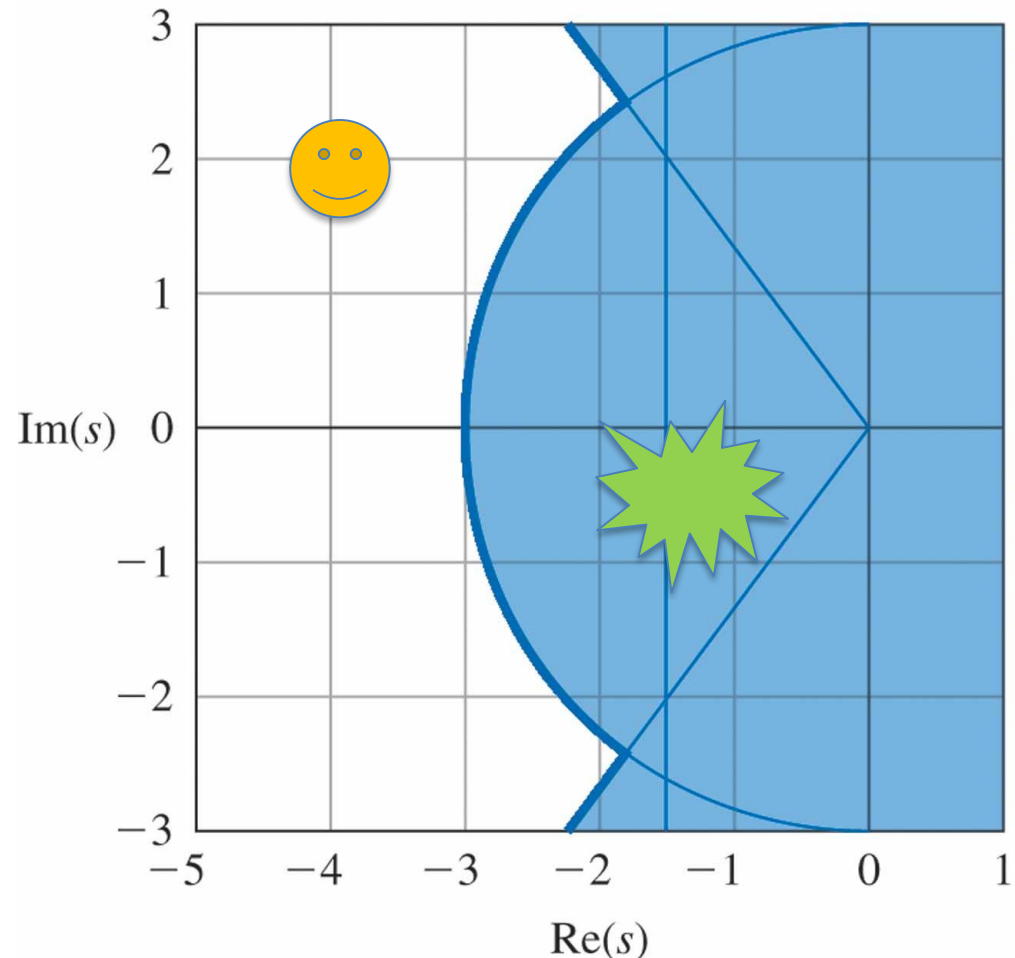
Solution:

$$\omega_n \geq \frac{1.8}{0.6} = 3 \text{ rad/sec}$$

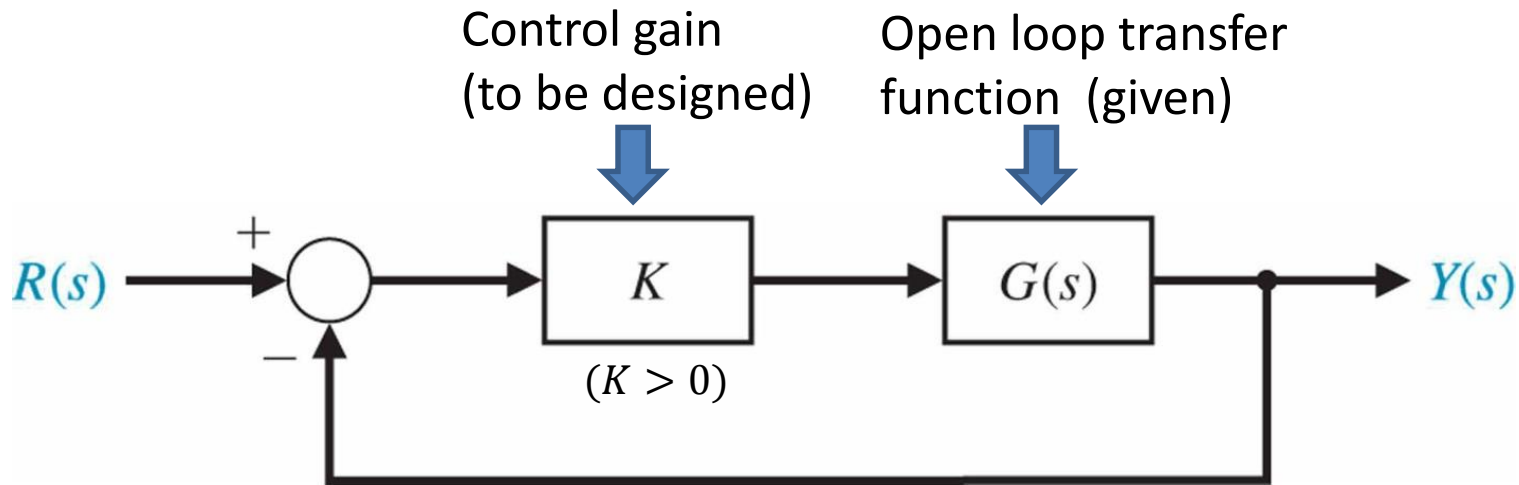
From Figure (M_p versus ζ)

$$\zeta \geq 0.6$$

$$\sigma \geq \frac{4.6}{x} = 1.5 \text{ sec}$$



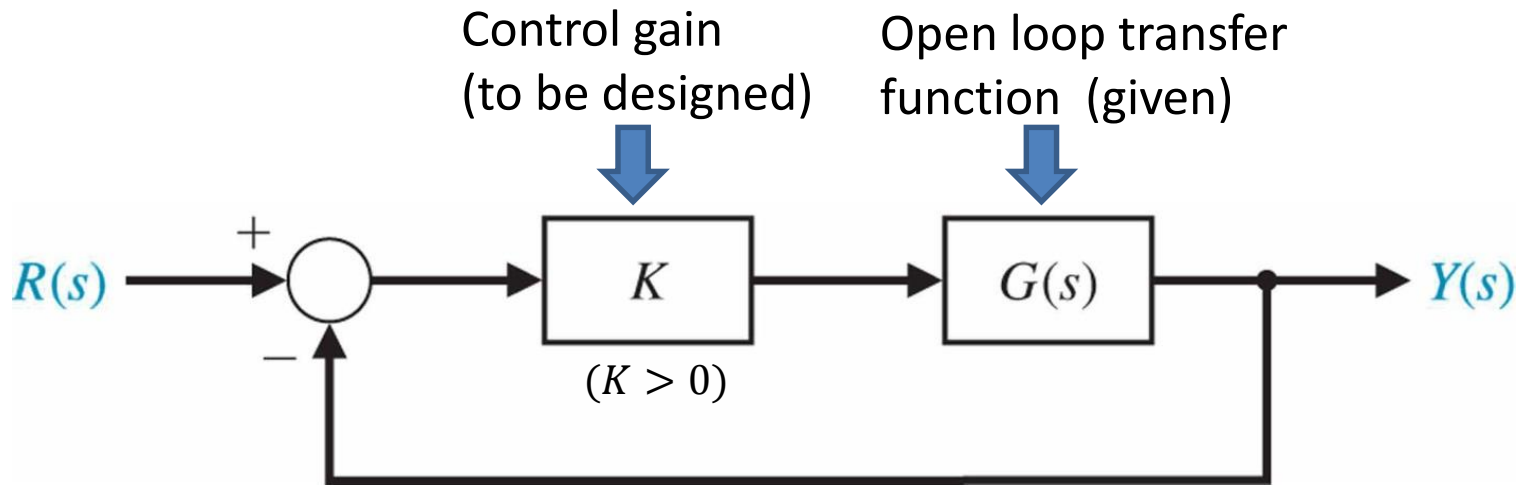
What is a Root Locus?



$$\frac{Y(s)}{R(s)} = H(s) = \frac{KG(s)}{1 + KG(s)}$$

- Characteristic equation: $D(s) = 1 + KG(s) = 0$
- Write $G(s) = \frac{b(s)}{a(s)} \rightarrow 1 + K \frac{b(s)}{a(s)} = 0 \rightarrow \frac{a(s)}{K} + b(s) = 0$

What is a Root Locus?



$$a(s) + Kb(s) = 0 \rightarrow a(s) \rightarrow 0 \text{ as } K \rightarrow 0$$

$$\frac{a(s)}{K} + b(s) = 0 \rightarrow b(s) \rightarrow 0 \text{ as } K \rightarrow \infty$$

Root Locus is the movement of the roots of

$$D(s) = 1 + KG(s) = 1 + K \frac{b(s)}{a(s)} \text{ as } K: 0 \rightarrow \infty$$

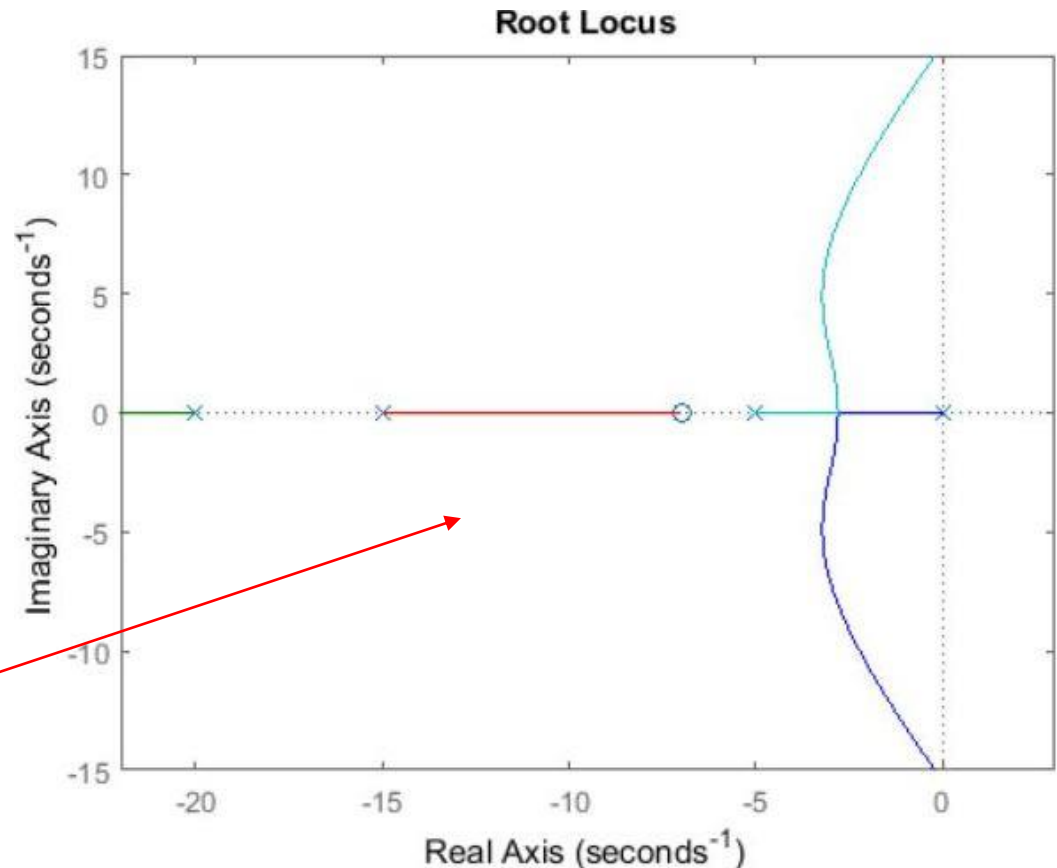
Root Locus of Transfer Function

Example: $\frac{Y(s)}{R(s)} = H(s) = \frac{s + 7}{s(s + 5)(s + 15)(s + 20)}$

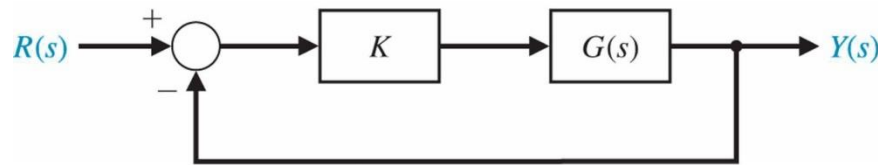
```
%MatLab code  
s = tf('s');  
sys = (s + 7)/(s*(s + 5)*(s + 15)*(s + 20));  
rlocus(sys)  
axis([-22 3 -15 15])
```

This MatLab plot shows
all possible closed-loop
pole locations for the
control gain $K: 0 \rightarrow \infty$

Acceptable Region
(left-hand side of curve)



Root Locus of Transfer Function



$$\frac{Y(s)}{R(s)} = \frac{s + 7}{s(s + 5)(s + 15)(s + 20)}$$

Now, **design** K such that

- overshoot less than 5% ($\rightarrow$ damping ratio $\zeta > 0.7$)
- rise time = 1 second ($\rightarrow \omega_n > 1.8$)

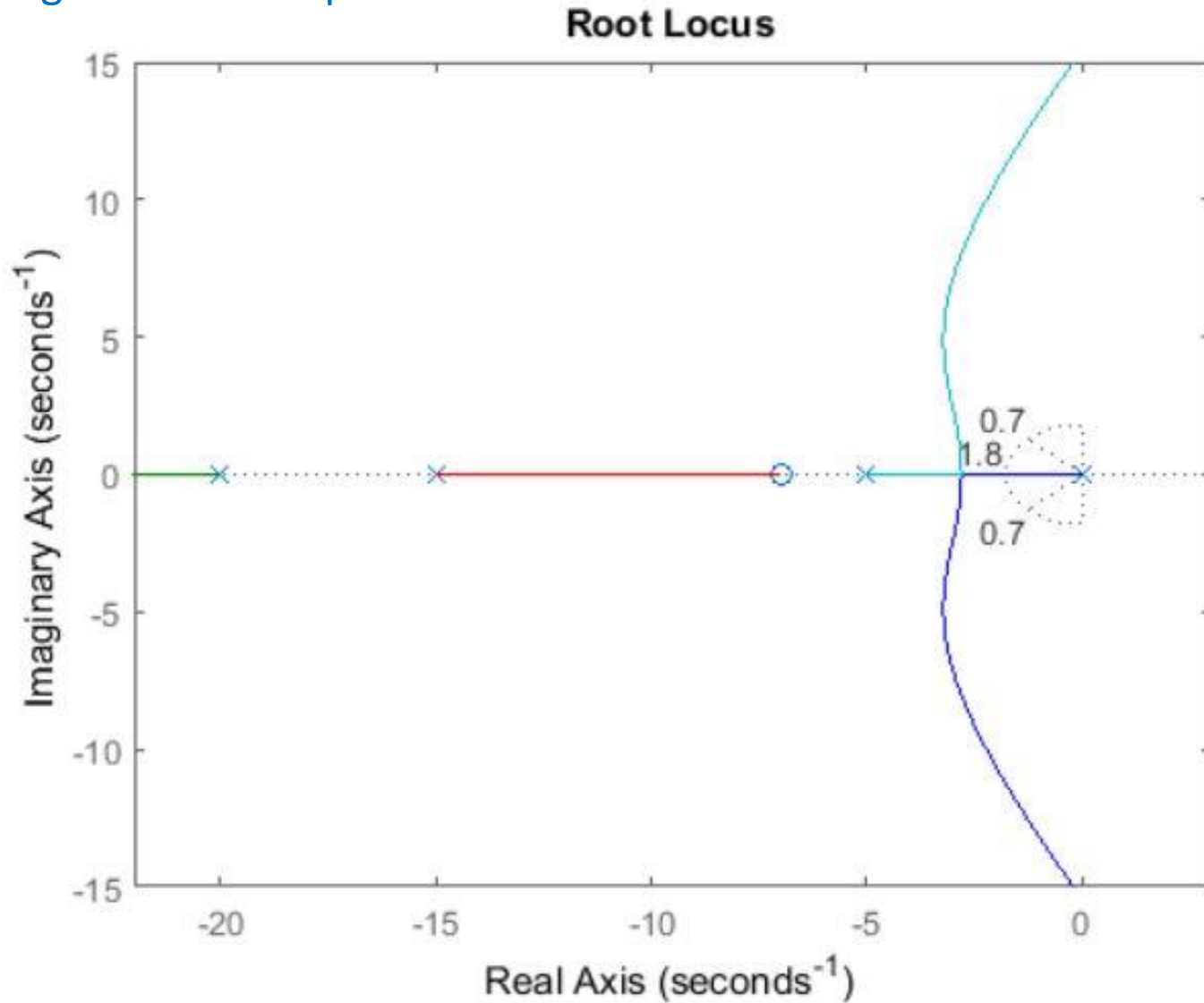
Not all closed-loop poles on root locus satisfy design requirements

To determine what part of the locus is acceptable, use MatLab:

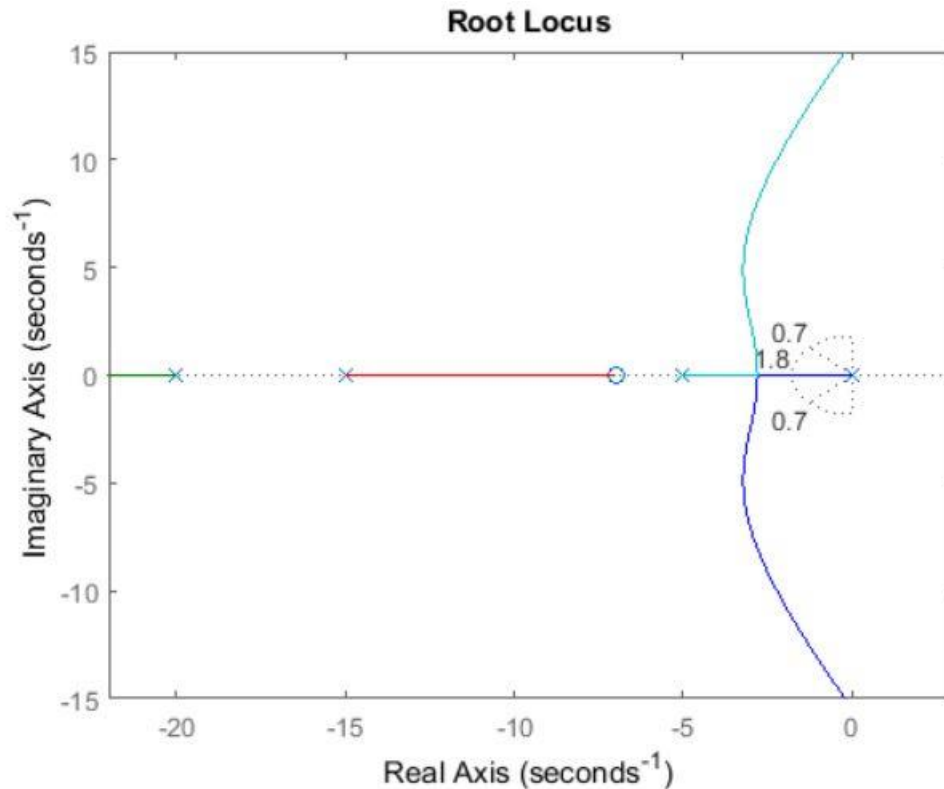
%MatLab code

```
s = tf('s');  
sys = (s + 7)/(s*(s + 5)*(s + 15)*(s + 20));  
rlocus(sys);  
axis([-22 3 -15 15]);  
zeta = 0.7;  
wn = 1.8;  
sgrid(zeta,wn)
```

MatLab generates this plot:

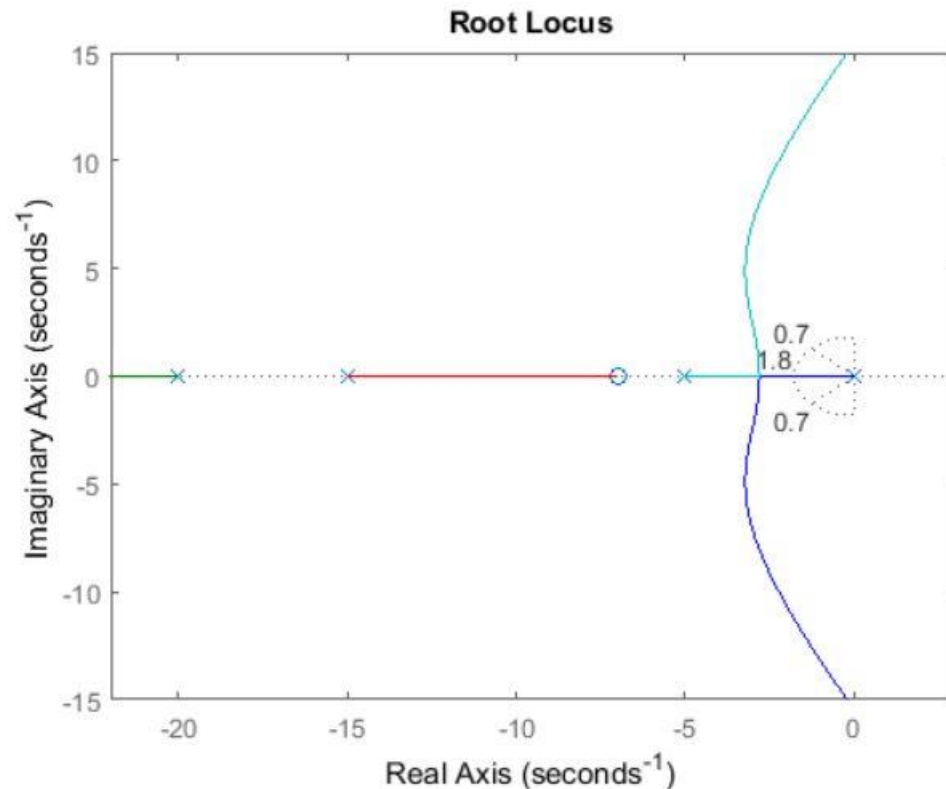


(Discussions on next two pages)



The two dotted lines at about 45-degree angle indicate pole locations with $\zeta = 0.7$; in between these lines, the poles have $\zeta > 0.7$ (**wanted**) and outside the two lines $\zeta < 0.7$

The semicircle indicates poles with $\omega_n = 1.8$; inside the circle, $\omega_n < 1.8$ and outside the circle $\omega_n > 1.8$ (**wanted**)



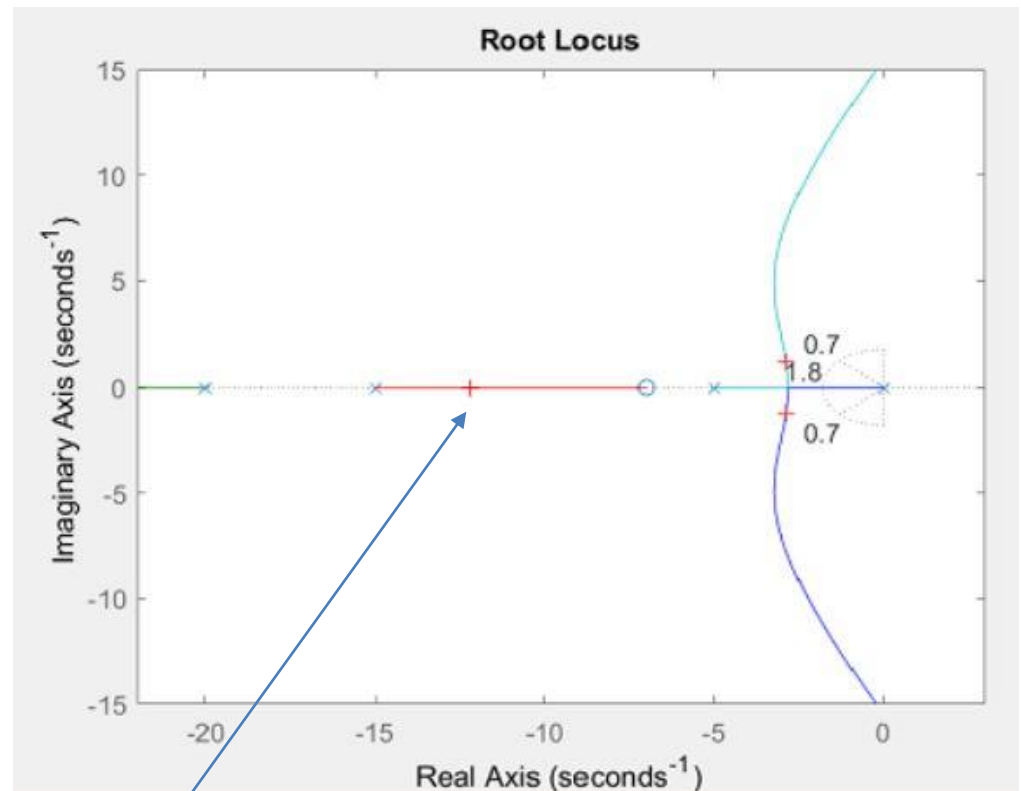
- For overshoot $M_p\% < 5\%$, poles should be in between the two dotted lines
- For rise time $t_r < 1$ second, poles should be outside the semicircle

All poles in the above locations are in LHP → closed-loop system is stable

Use MATLAB to choose the desired poles on the locus:

`[K,poles] = rlocfind(sys)` (See next page)

```
%MatLab code
s = tf('s');
sys = (s + 7)/(s*(s + 5)*(s + 15)*(s + 20));
rlocus(sys);
axis([-22 3 -15 15]);
zeta = 0.7;
wn = 1.8;
sgrid(zeta,wn);
[K,poles] = rlocfind(sys)
```



Click at the point where you want the closed-loop pole to be
(for example, the one marked by "+")

You may also select some point within the allowable region,
which are not unique but affect response (Recall: Lecture-3)

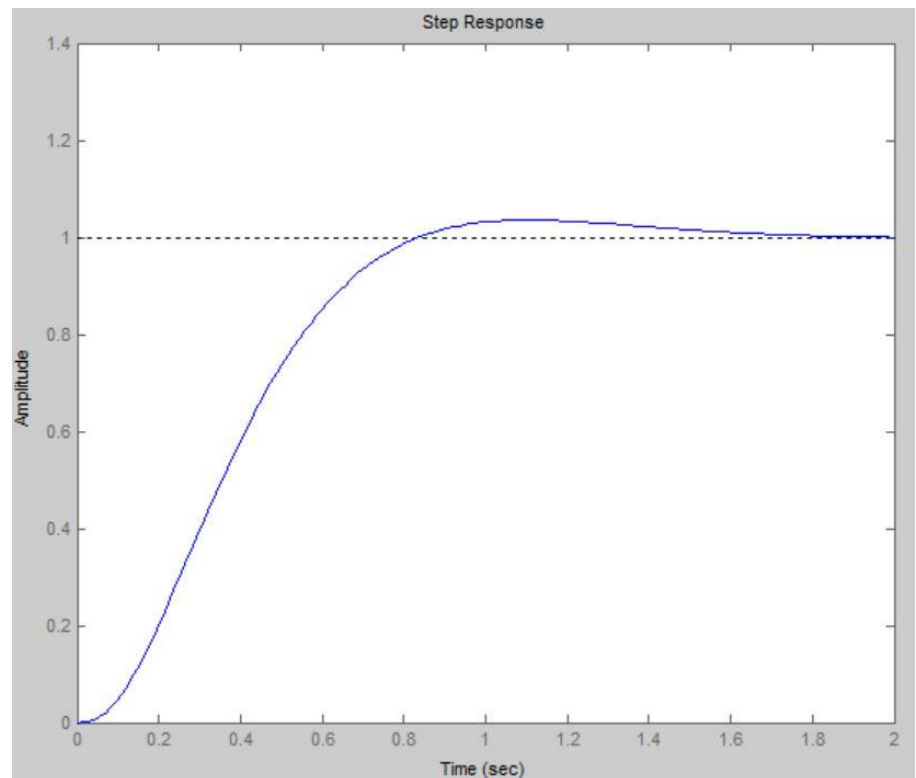
To verify the Step Response:

Use MatLab

%MatLab code

```
s = tf('s');  
sys = (s + 7)/(s*(s + 5)*(s + 15)*(s + 20));  
rlocus(sys);  
axis([-22 3 -15 15]);  
zeta = 0.7;  
wn = 1.8;  
sgrid(zeta,wn);  
K = 600;  
sys_cl = feedback(K*sys,1)  
step(sys_cl)
```

The response has
overshoot $M_p \% < 5\%$
and rise time $t_r < 1$
second



Remark:

Use MatLab (no need to enter K value if using command “rlocfind”):

%MatLab code

```
s = tf('s');
```

```
sys = (s + 7)/(s*(s + 5)*(s + 15)*(s + 20));
```

```
rlocus(sys);
```

```
axis([-22 3 -15 15]);
```

```
zeta = 0.7;
```

```
wn = 1.8;
```

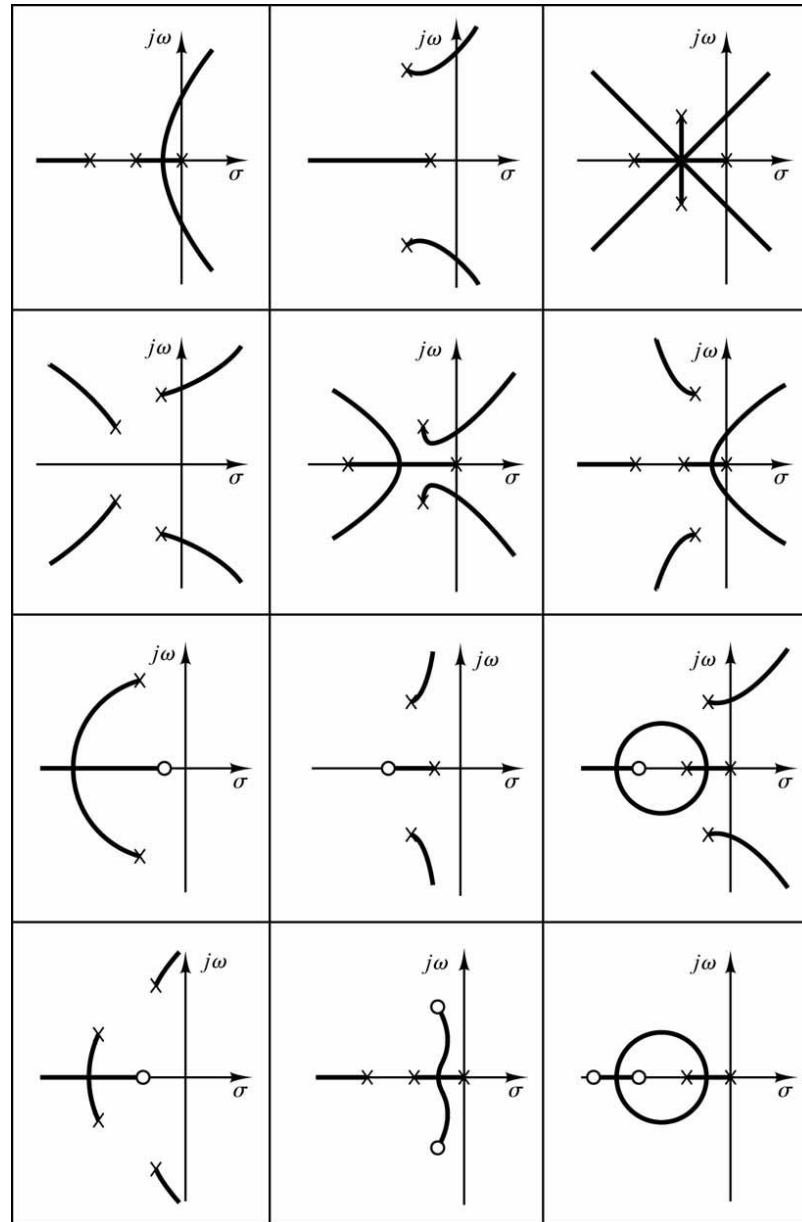
```
sgrid(zeta,wn);
```

```
[K,poles] = rlocfind(sys); %No need to enter value, e.g., K=600
```

```
sys_cl = feedback(K*sys,1)
```

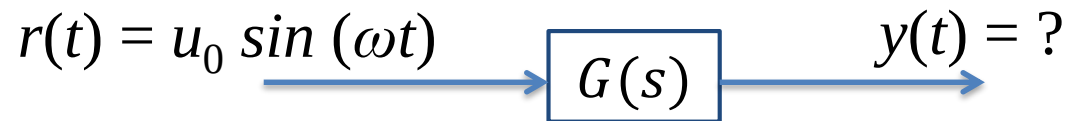
```
step(sys_cl)
```

Typical Pole–Zero Locations and their Root Loci



More about Root Locus

- Design using $G(j\omega)$, not $G(s)$, with root locus analysis
- Transfer Function for sinusoidal input of frequency ω



- $$Y(s) = G(s)R(s) = \frac{C \prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)} \frac{u_0 \omega}{s^2 + \omega^2}$$

where $\{p_i\}$ and $\{z_i\}$ are poles and zeros of $G(s)$

- Partial fraction expansion and Inverse Laplace Transform →

$$y(t) = A_1 e^{P_1 t} + \dots + A_n e^{P_n t} + A_0 e^{j\omega t} + A_0^* e^{-j\omega t}$$

where $A_0 = (s - j\omega)Y(s)|_{s=j\omega} = G(j\omega)u_0/(2j)$

-

$$y(t) = A_1 e^{P_1 t} + \dots + A_n e^{P_n t} + |G(j\omega)| u_0 \sin(\omega t + \angle G(j\omega))$$

- If the poles of $G(s)$ are distinct and located in LHP, then:

For large t , $y_{ss}(t) = |G(j\omega)| u_0 \sin(\omega t + \angle G(j\omega))$

- Comparing $y_{ss}(t)$ to $r(t) = u_0 \sin(\omega t)$

$|G(j\omega)|$ is the amplitude **scaling factor** of $y_{ss}(t)$ over $r(t)$

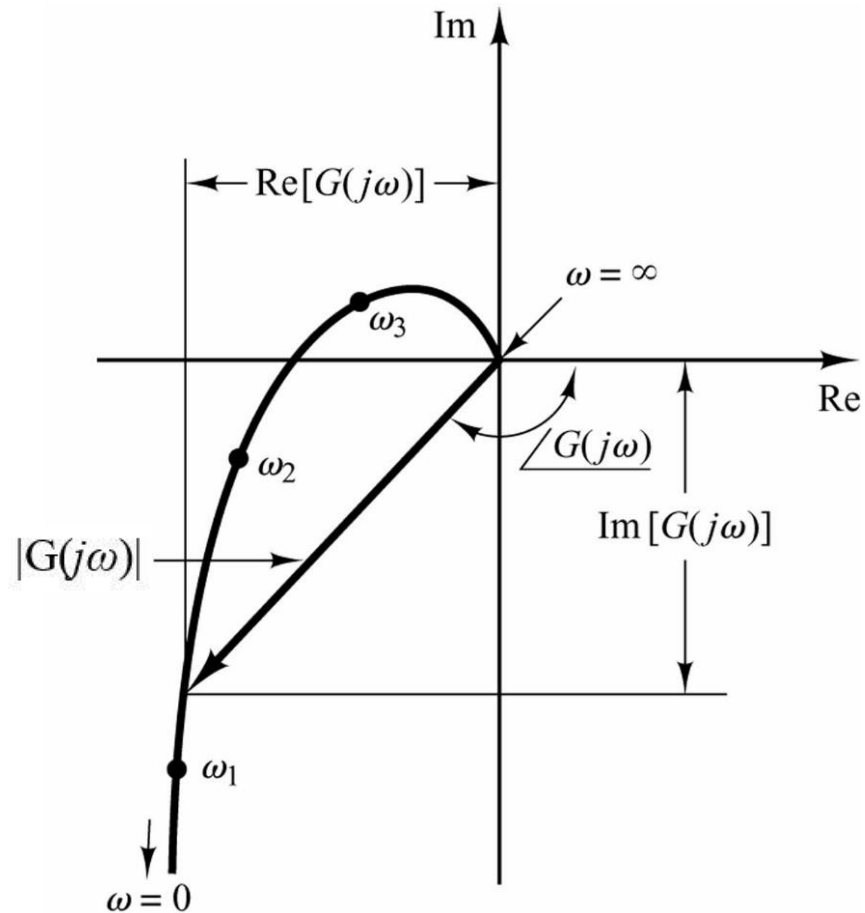
$\angle G(j\omega)$ is the **additional phase (angle)** of $y_{ss}(t)$ over $r(t)$

Representation of $G(j\omega)$

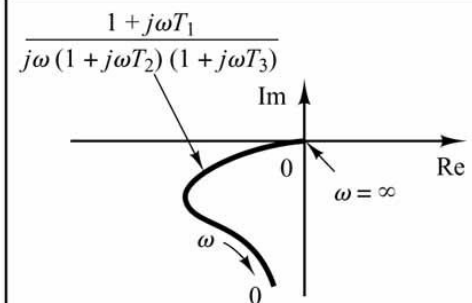
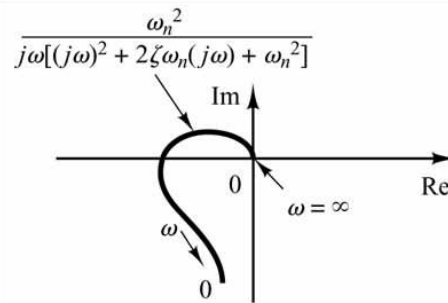
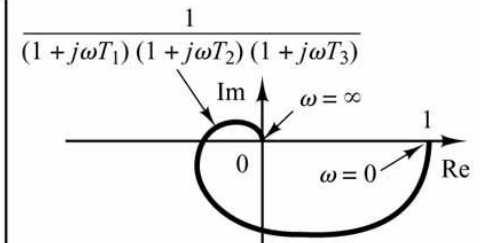
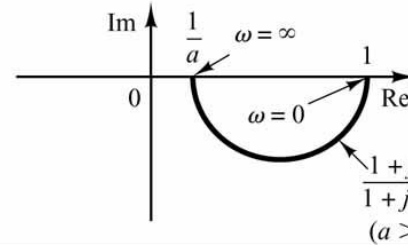
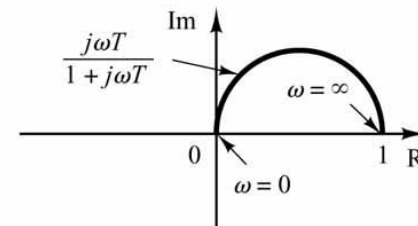
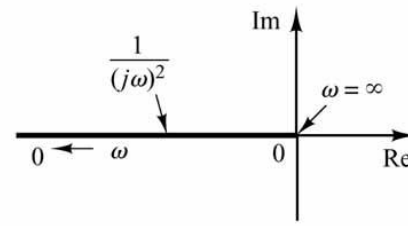
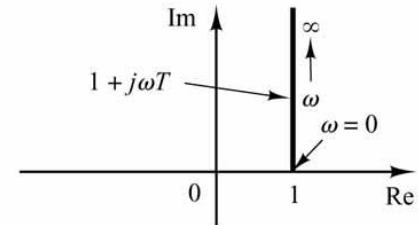
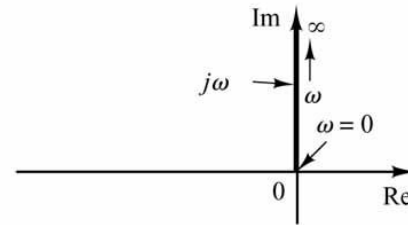
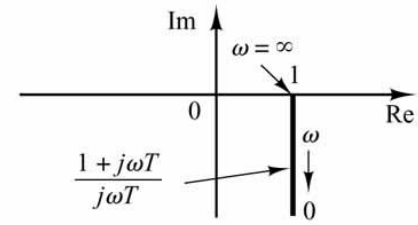
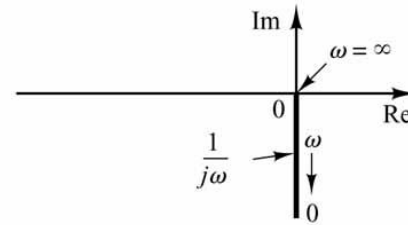
Polar Plot

All information (magnitude, phase, frequency) in **one** plot

$$G(s) = \frac{C \prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)}$$



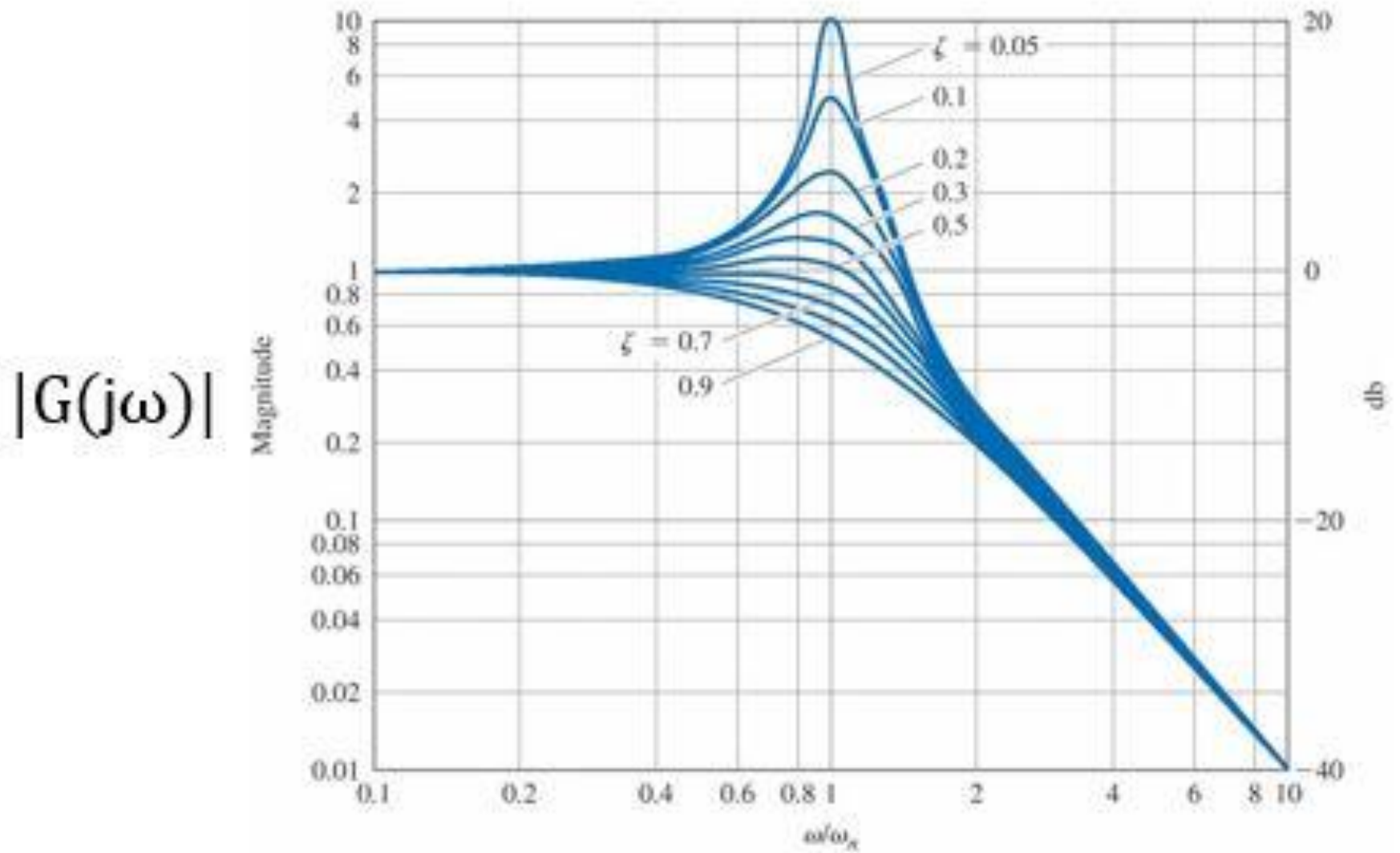
Polar Plots of some simple Transfer Functions



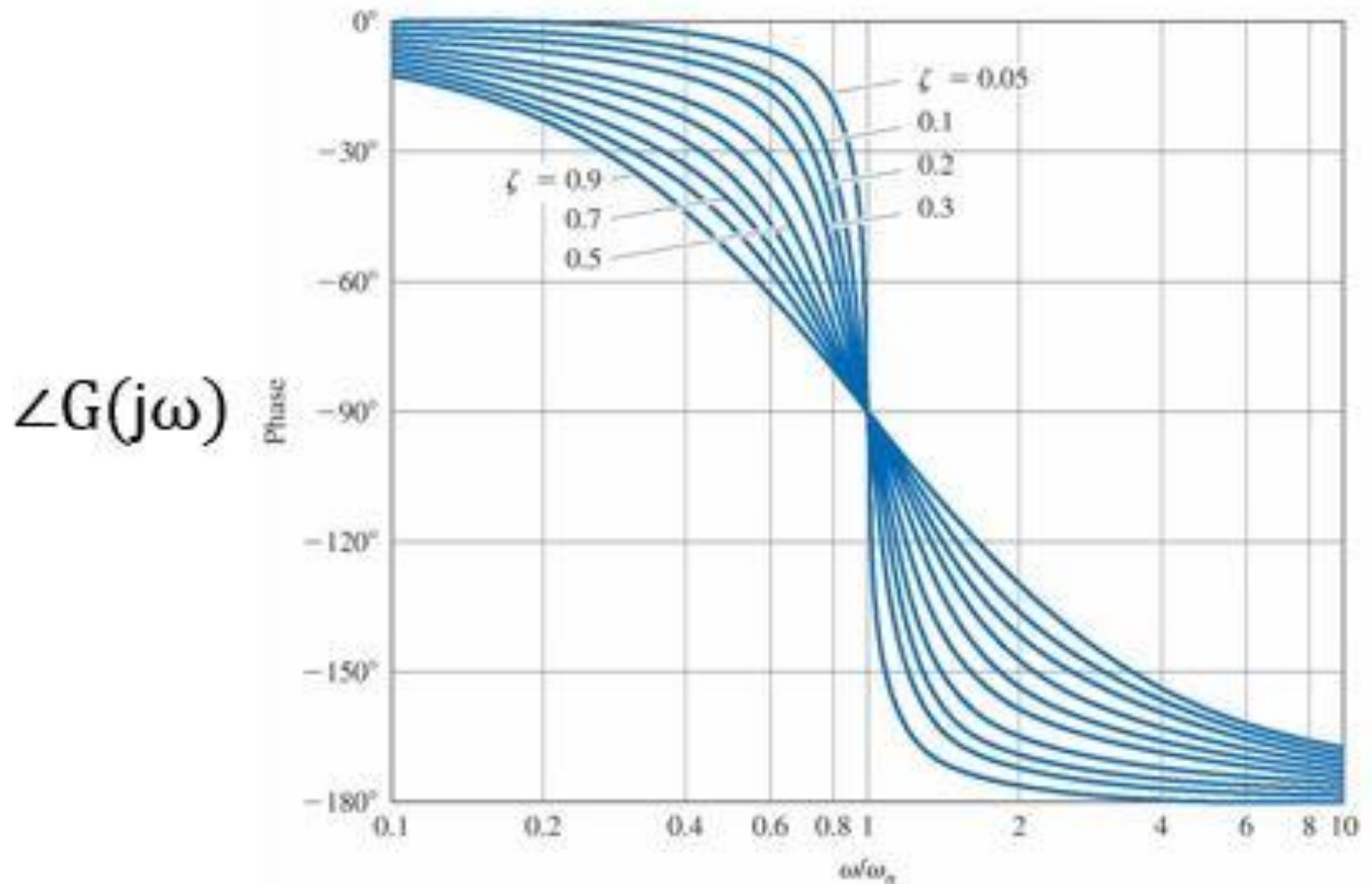
Bode Plots

- Present the information in **2** plots:
 - i) Magnitude: $\log_{10} |G(j\omega)|$ **vs** $\log_{10} \omega$
 - ii) Phase: $\angle G(j\omega)$ **vs** $\log_{10} \omega$
- Advantages of log-log scale:
 - For ω axis to cover a wide frequency range
 - For $G(j\omega) = G_1(j\omega)G_2(j\omega)$
 - $\rightarrow \log_{10} |G(j\omega)| = \log_{10} |G_1(j\omega)| + \log_{10} |G_2(j\omega)|$
 - $\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$
 - $\rightarrow$ Bode plots of G
 - $=$ Bode plots of G_1 + Bode plots of G_2
- Special dB unit for magnitude plot:
$$|G(j\omega)|_{\text{dB}} = 20 \log_{10} |G(j\omega)|$$

Bode Plot of Transfer Function of Standard Second-Order System



Bode Plot of Transfer Function of Standard Second-Order System



Minimum-Phase Systems and Nonminimum-Phase Systems

Minimum-Phase Transfer Functions

Minimum-Phase Systems

They have **no** poles and **no** zeros inside RHP

$$G_a(s) = \frac{1+s}{1+2s}$$

Nonminimum-Phase Transfer Functions

Nonminimum-Phase Systems

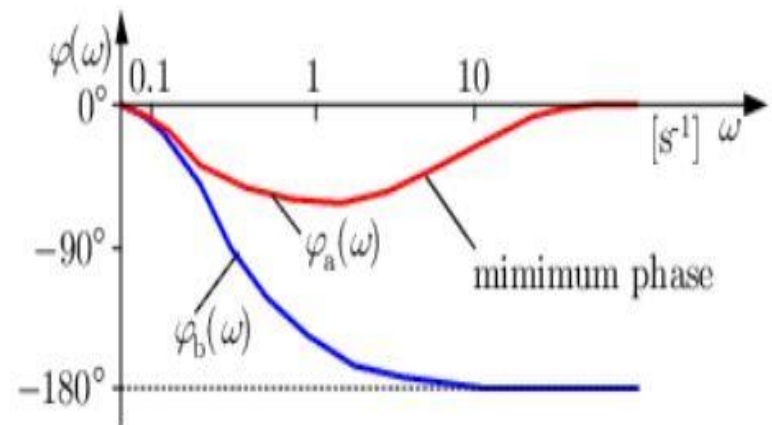
They have **either** poles **or** zeros inside RHP

$$G_b(s) = \frac{1-s}{1+2s}$$

Magnitude: $|G_a(s)| = |G_b(s)|$

Phase:

$$\varphi_a = -\tan^{-1} \frac{\omega}{1+2\omega^2}$$
$$\varphi_b = -\tan^{-1} \frac{\omega}{1-2\omega^2}$$



Example:

Draw the Bode plot of the Transfer Function

$$G(j\omega) = \frac{e^{-j\omega L}}{1+j\omega T}$$

Solution:

Log-magnitude:

$$\begin{aligned} 20 \log|G(j\omega)| &= 20 \log|e^{-j\omega L}| + 20 \log\left|\frac{1}{1+j\omega T}\right| \\ &= 0 + 20 \log\left|\frac{1}{1+j\omega T}\right| \end{aligned}$$

Phase angle:

$$\begin{aligned} \angle G(j\omega) &= \angle e^{-j\omega L} + \angle \frac{1}{1+j\omega T} \\ &= -\omega L - \tan^{-1} \omega T \end{aligned}$$

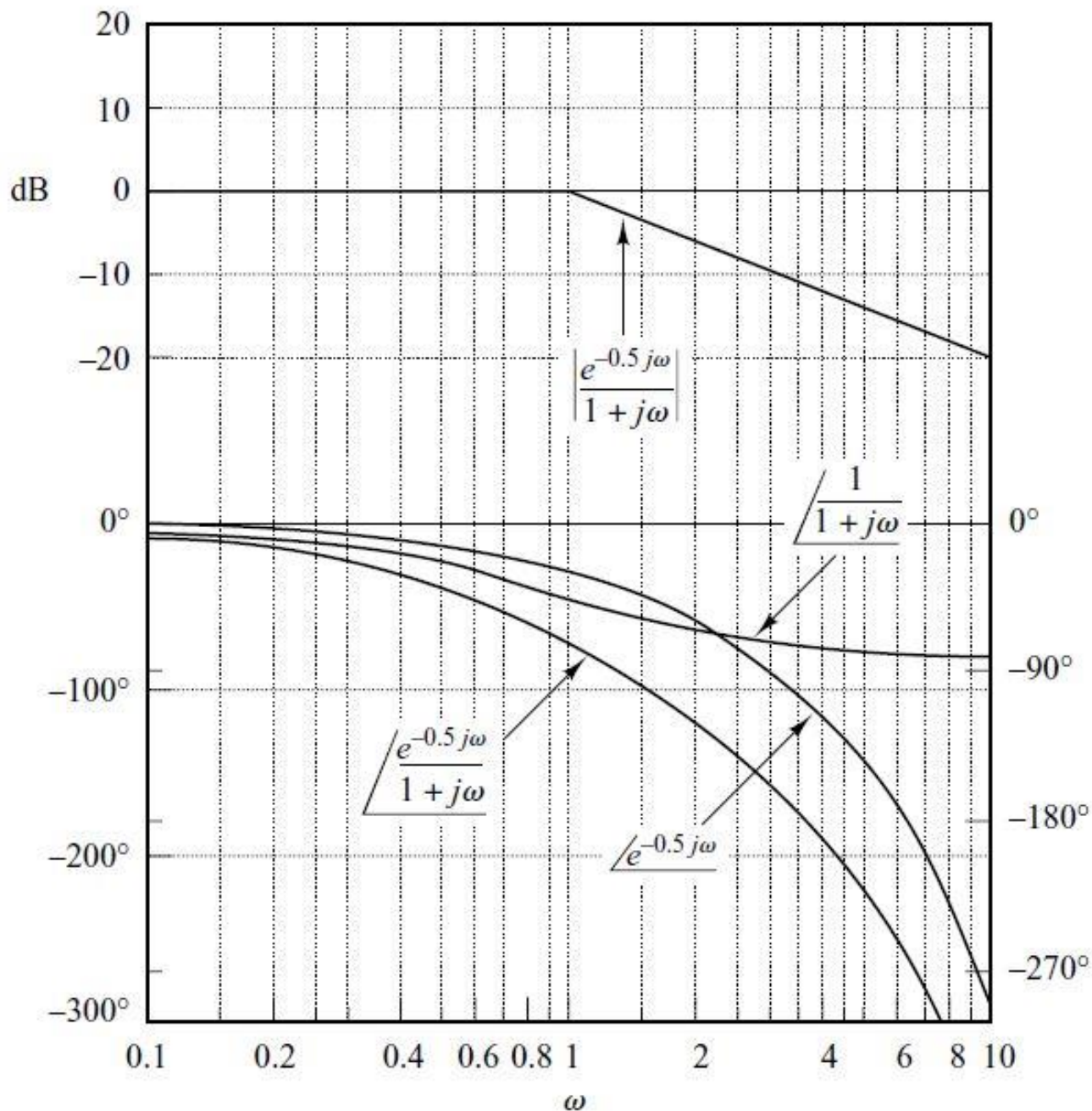
$$G(j\omega) = \frac{e^{-j\omega L}}{1 + j\omega T}$$

$$L = 0.5$$

$$T = 1$$



$$G(j\omega) = \frac{e^{-0.5j\omega}}{1 + j\omega}$$

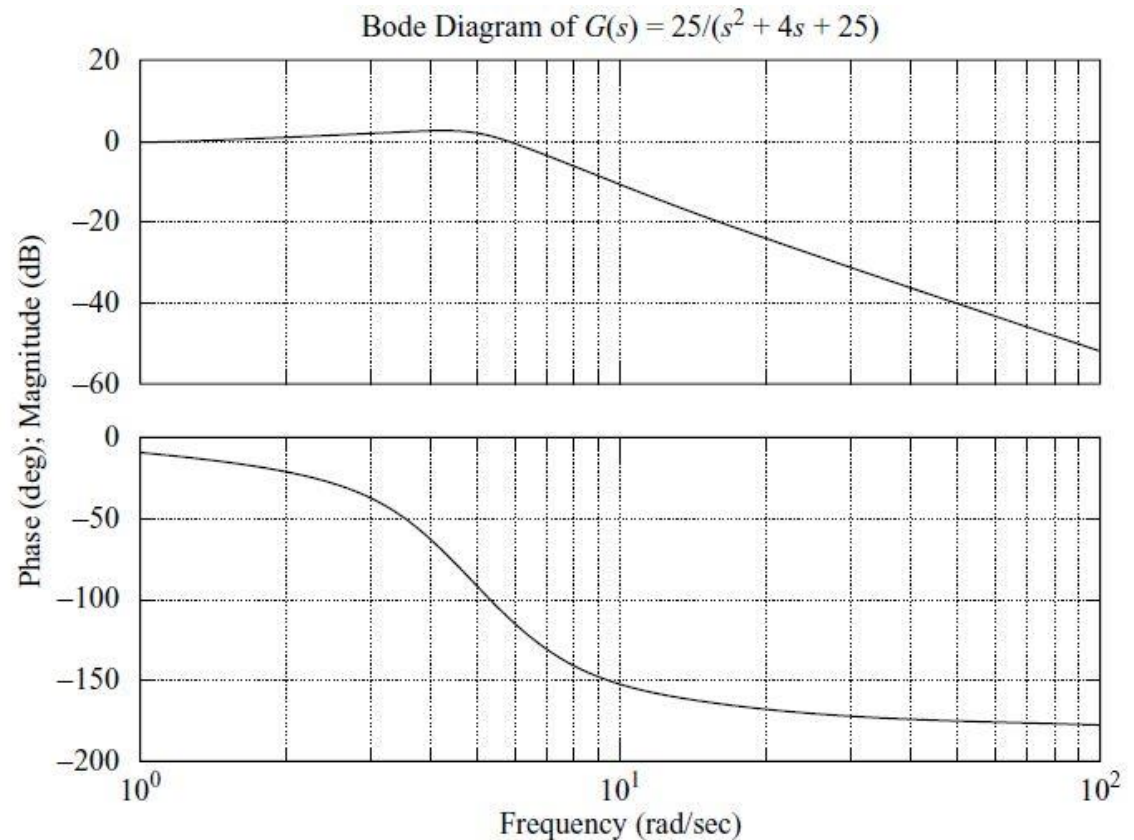


Code Plots by MatLab

Example:

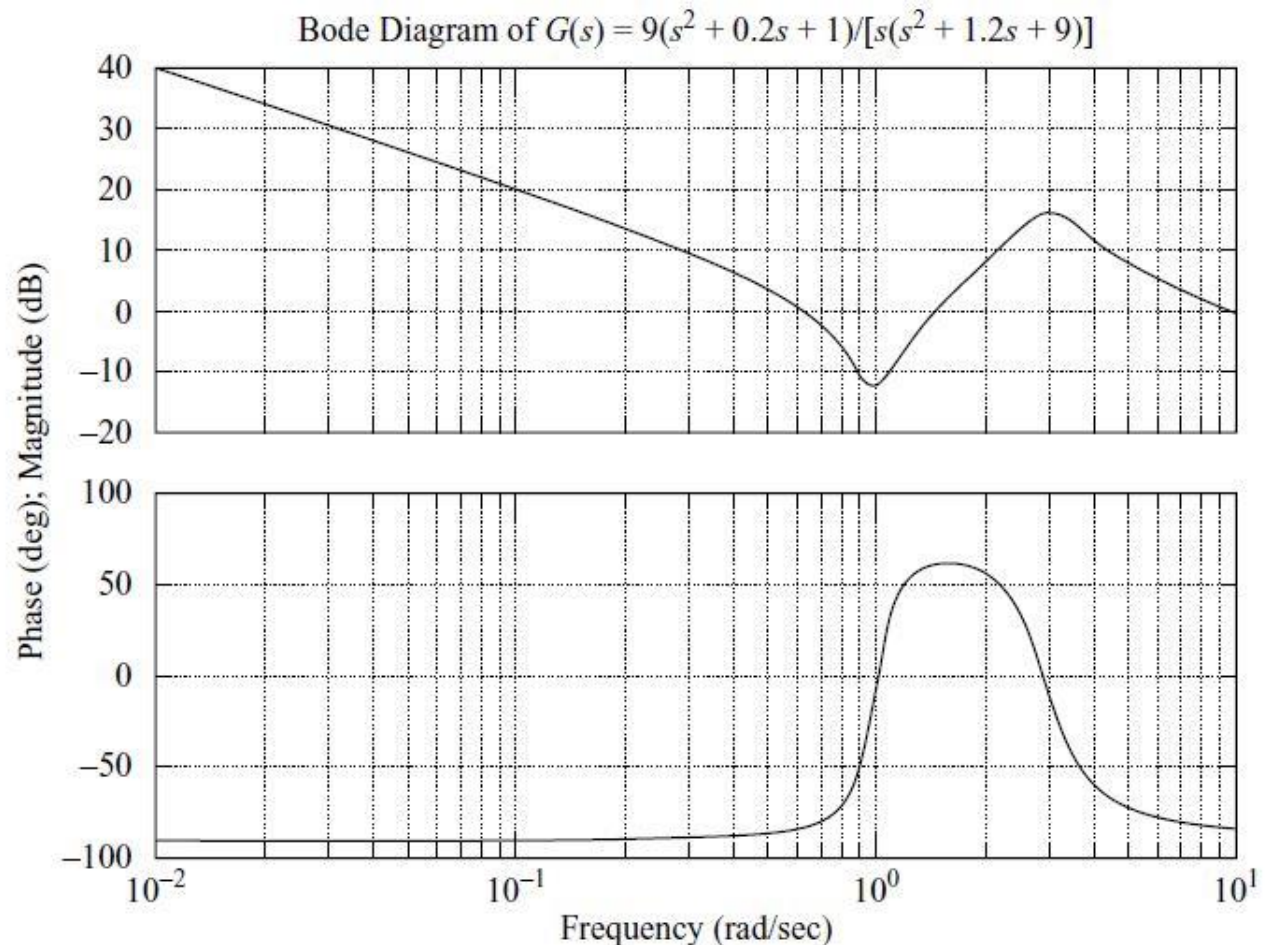
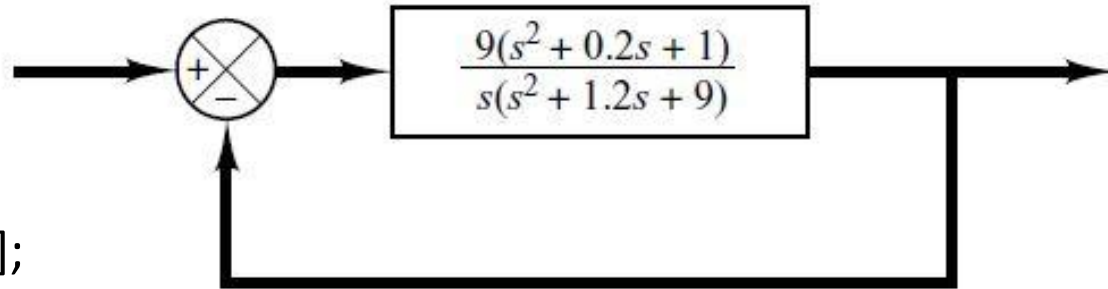
Transfer Function $G(s) = \frac{25}{s^2 + 4s + 25}$

```
%MatLab code  
num = [25];  
den = [1 4 25];  
bode(num,den);  
grid
```

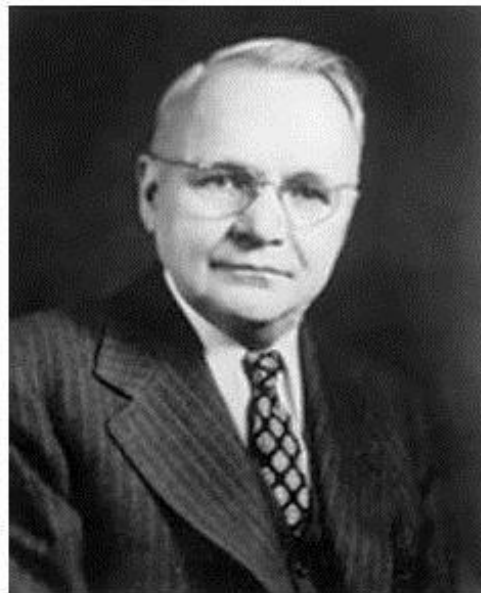
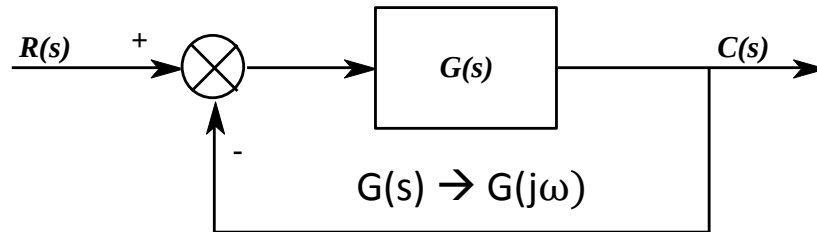


Example:

```
%MatLab code  
num = [9 1.8 9];  
den = [1 1.2 9 0];  
bode(num,den);  
grid
```

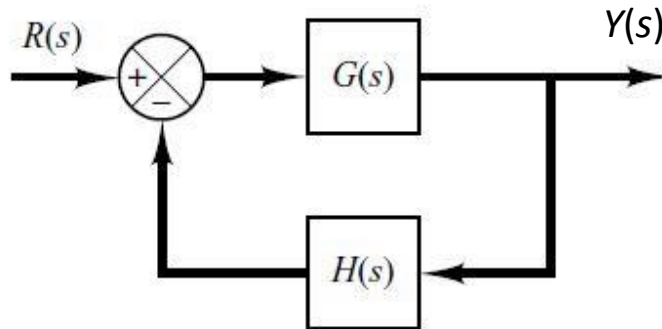


Stability Analysis with $G(j\omega)$



Harry Nyquist
(1889 – 1976)

Stability Analysis with $G(j\omega)$



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Assumption:

$G(s)H(s)$ has no poles or zeros on imaginary axis

Closed-loop system stability if and only if all poles are located on the LHP

Nyquist Stability Criterion:

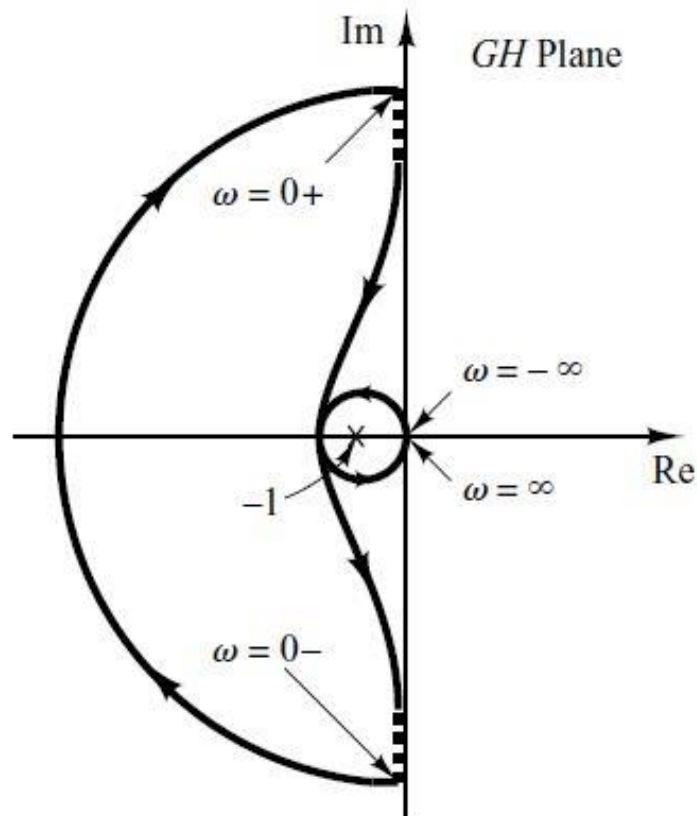
If (i) the Transfer Function $G(s)H(s)$ has P poles in the RHP and (ii) $\lim_{s \rightarrow \infty} G(s)H(s) = \text{constant}$, then for closed-loop stability, as $\omega: -\infty \rightarrow +\infty$, $G(j\omega)H(j\omega)$ locus must encircle the point $(-1 + j0)$ for P times counterclockwise

Example: $G(s)H(s) = \frac{K(s+3)}{s(s-1)} \quad (K > 0)$

(i) Transfer Function $G(s)H(s)$ has one pole ($s = 1$) on RHP ($P = 1$)

(ii) $\lim_{s \rightarrow \infty} G(s)H(s) = 0$ (constant)

Nyquist Plot →



Point $-1 + 0j$ is encircled by the curve once ($P = 1$) counterclockwise



Closed-loop system $\frac{G(s)}{1+G(s)H(s)}$ is stable

Recall: Nyquist Stability Criterion:

If: (i) Transfer Function $G(s)H(s)$ has P poles in RHP; (ii) $\lim_{s \rightarrow \infty} G(s)H(s) = \text{constant}$, then for closed-loop stability, as $\omega: -\infty \rightarrow +\infty$, $G(j\omega)H(j\omega)$ locus must encircle the point $(-1 + j0)$ for P times **counterclockwise**

Equivalently --

Nyquist Stability Criterion:

The above criterion can be expressed as

$N + P = 0$ (this must hold for the closed-loop system to be stable)

N = number of **clockwise** encirclements of the point $-1+j0$

($-N$ = number of **counterclockwise** encirclements)

P = number of poles of $G(s)H(s)$ in RHP

(In fact, there is a relationship: $N + P = Z$)

Z = number of zeros of $1+G(s)H(s)$ in RHP)

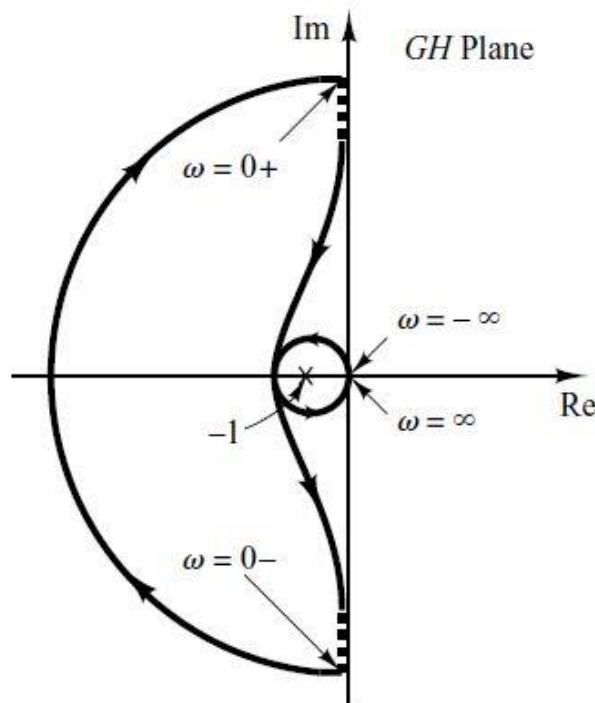
→ For the closed-loop system to be stable:

$1+G(s)H(s)$ must have no zeros in RHP

Example: $G(s)H(s) = \frac{K(s+3)}{s(s-1)} \quad (K > 1)$

- (i) Transfer Function $G(s)H(s)$ has one pole ($s = 1$) on RHP ($P = 1$)
- (ii) $\lim_{s \rightarrow \infty} G(s)H(s) = 0$ (constant)

Nyquist Plot →



Point $-1 + 0j$ is encircled by curve **counterclockwise** (so, $N = -1$)

→ $N + P = 0$ →

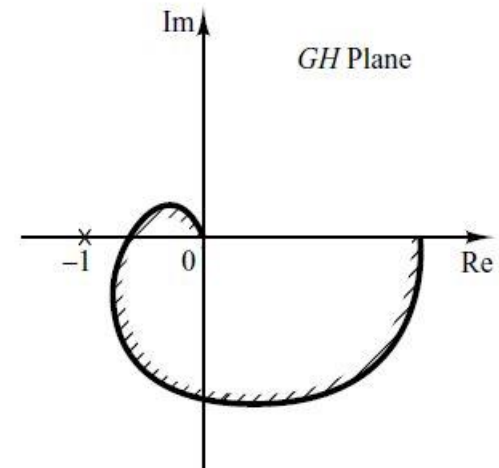
Closed-loop system $\frac{G(s)}{1+G(s)H(s)}$ is stable
(same as before)

In fact, Transfer Function $1 + G(s)H(s) = \frac{s^2 + (K-1)s + 3K}{s(s-1)}$ has **no** zero on RHP ($Z = 0$)

Remarks

If $P \neq 0$, for a closed-loop stable system, one must have $Z = 0$, therefore $N = -P \rightarrow$ There must be P **counterclockwise** encirclements of point $-1 + j0$

If $P = 0$, then $Z = N \rightarrow$ for stability, there must be **no encirclement** of point $-1 + j0$ by the $G(j\omega)H(j\omega)$ locus $\rightarrow$ For stability, the point $-1 + j0$ must be located **outside** the shaded region:



Example:

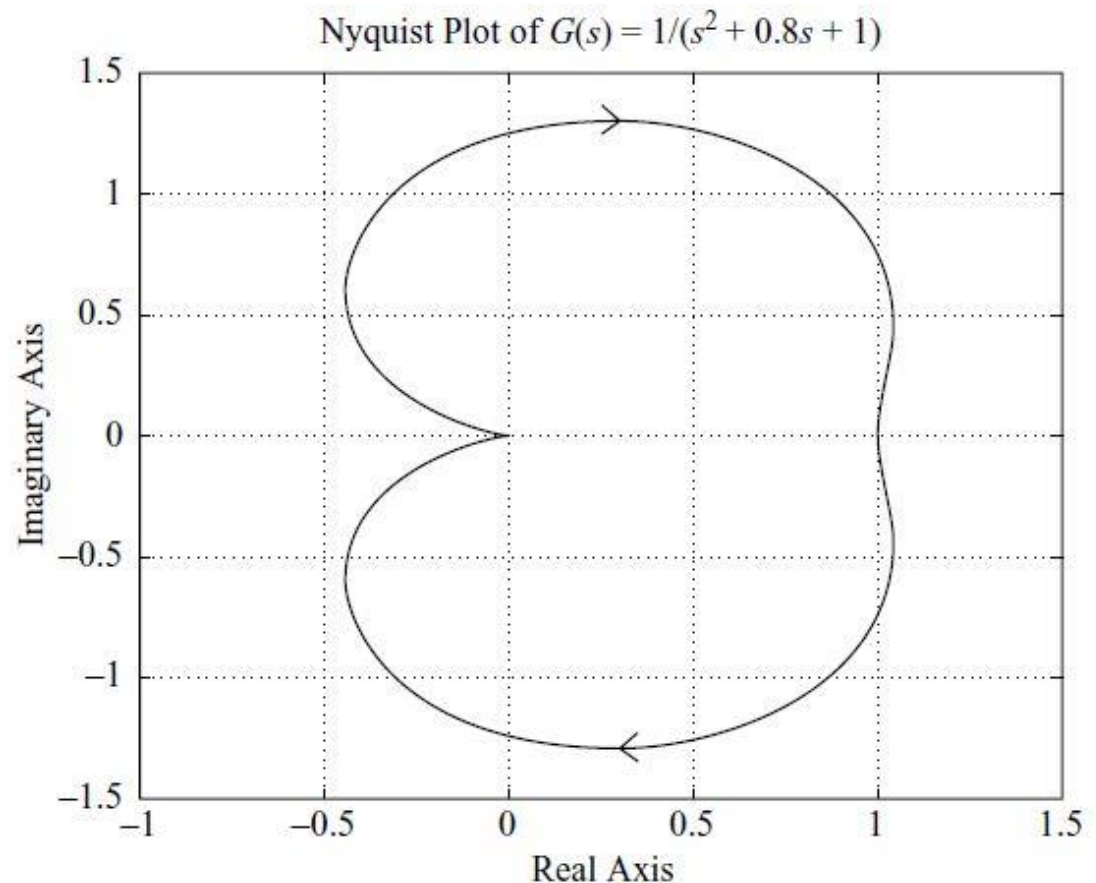
Transfer Function $G(s)H(s) = \frac{1}{s^2 + 0.8s + 1}$

There are no poles of $G(s)H(s)$ in the RHP, so $P = 0$, and the point $-1 + j0$ is not encircled, so $N = 0$,

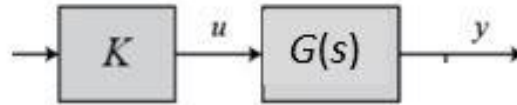
→ $Z = 0 = N + P$

→ Closed-loop system is stable

```
%MatLab code  
num = [1];  
den = [1 0.8 1];  
nyquist(num,den)
```

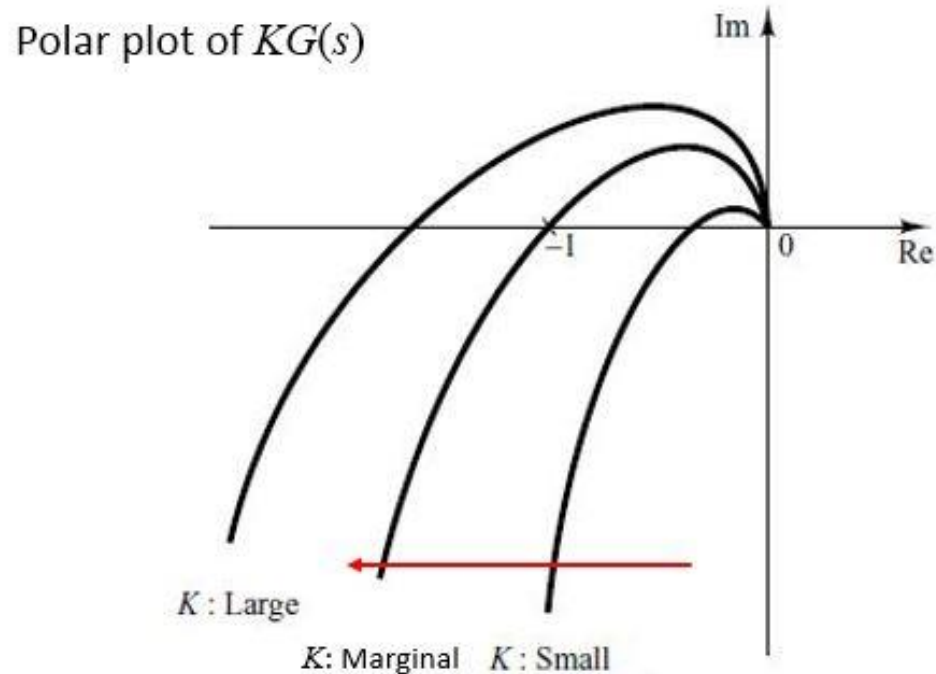


Common Case:



If the $KG(j\omega)$ locus **passes through** the point $-1 + j0$, then closed-loop poles are located on the imaginary axis

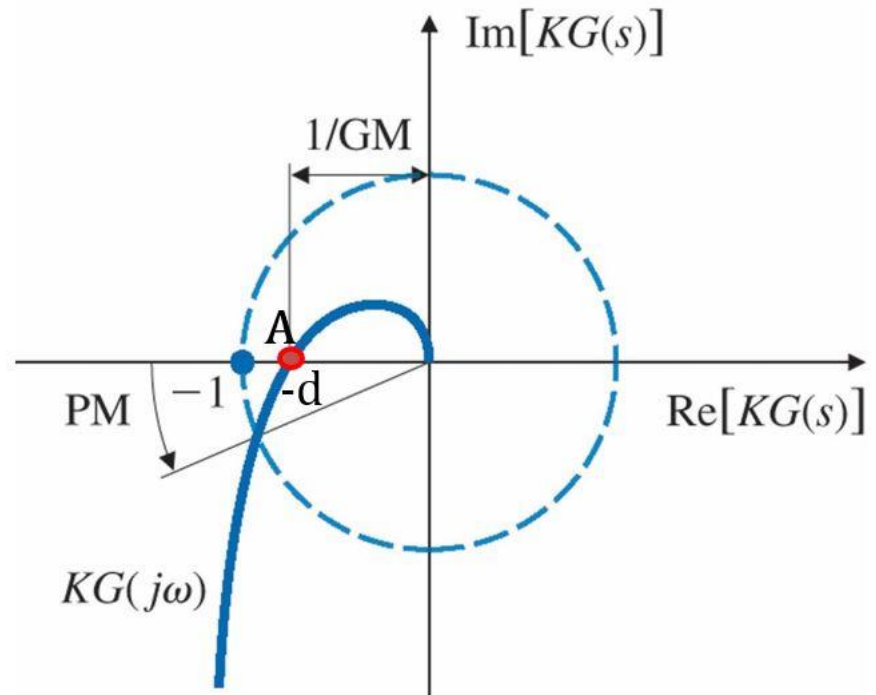
This is **marginally stable** and is not desirable for practical control systems, so it should be avoided by appropriately designing the control gain K



Gain Margin

- **Definition:** Gain margin is the maximum scalar factor multiplying to $KG(j\omega)$ for which polar plot is still not encircling $(-1,0)$
- Let polar plot cut real axis at point A $(-d, 0)$, $0 < d < 1$
- Gain Margin (GM):

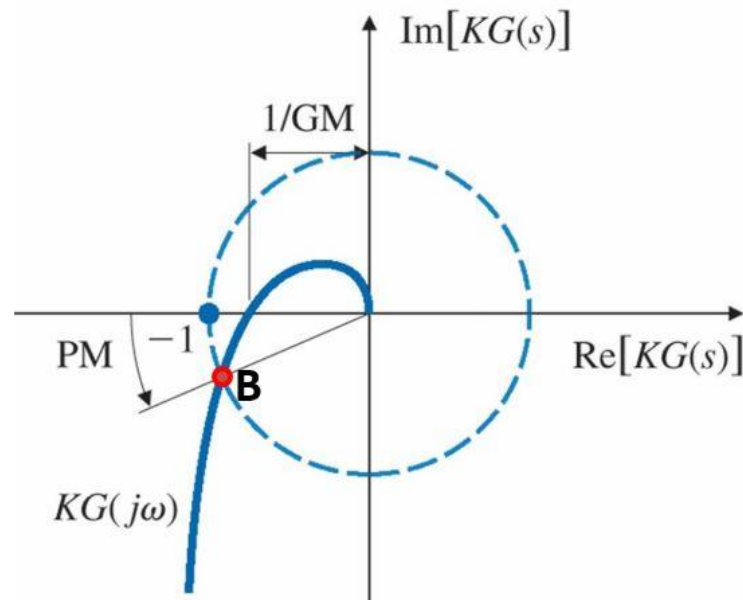
$$GM = \frac{1}{d}$$



Phase Margin

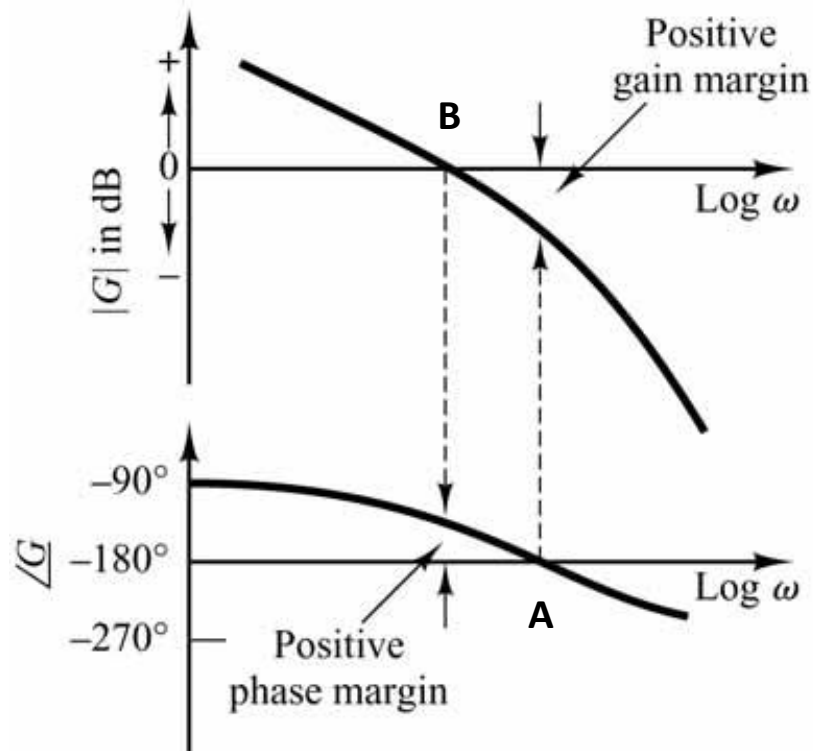
- **Definition:** Phase margin is the maximum **clockwise** rotation of polar plot (about the origin), which is still not encircling $(-1,0)$
- Draw a circle of radius 1 about the origin and mark the intersection point as point B
- Phase Margin (PM) = $\angle G(j\omega_B) - (-180^\circ)$

(PM $\cong 100 \zeta$)



GM and PM from Bode Plot

- Point A occurs at point where phase = -180°
- Point B occurs at point where magnitude = 1 = 0 dB



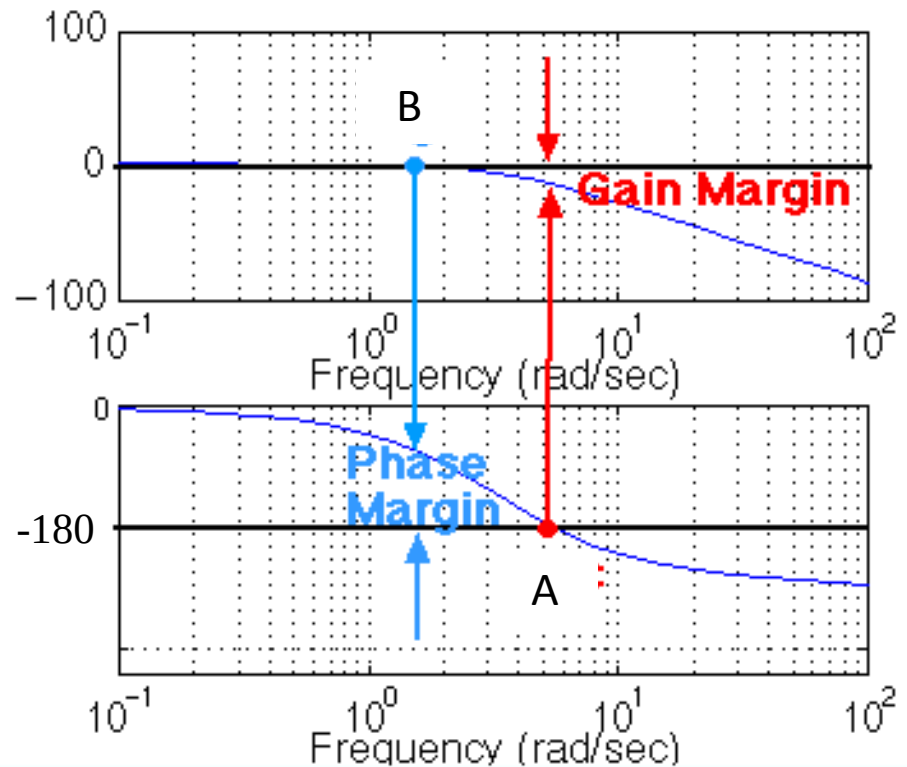
GM and PM from Bode Plot

Example: $50/(s^3 + 9s^2 + 30s + 40)$

%MatLab code
`bode(50, [1 9 30 40])`

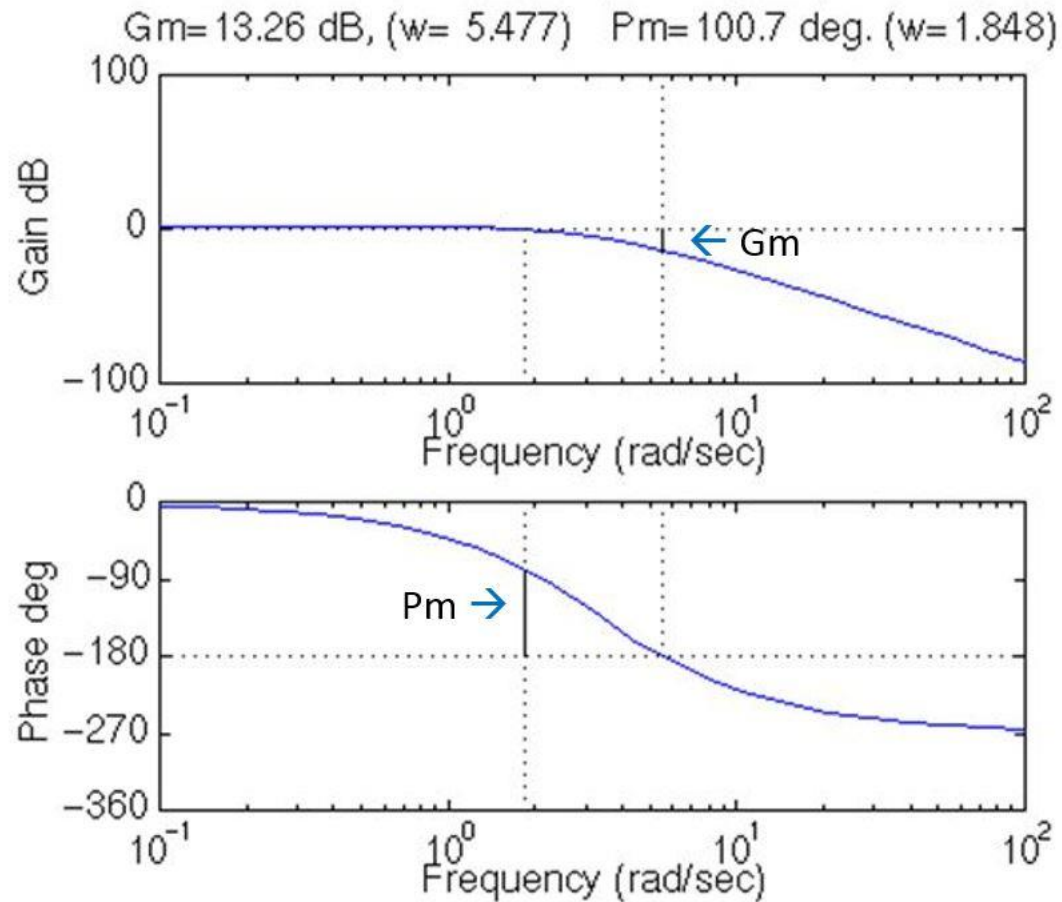
Gain Margin (GM) is the difference between the magnitude curve and 0dB at the point corresponding to the phase of -180 degree

Phase Margin (PM) is the difference between the phase curve and -180 degree at the point corresponding to the gain of 0dB



%MatLab code
margin(50,[1 9 30 40])

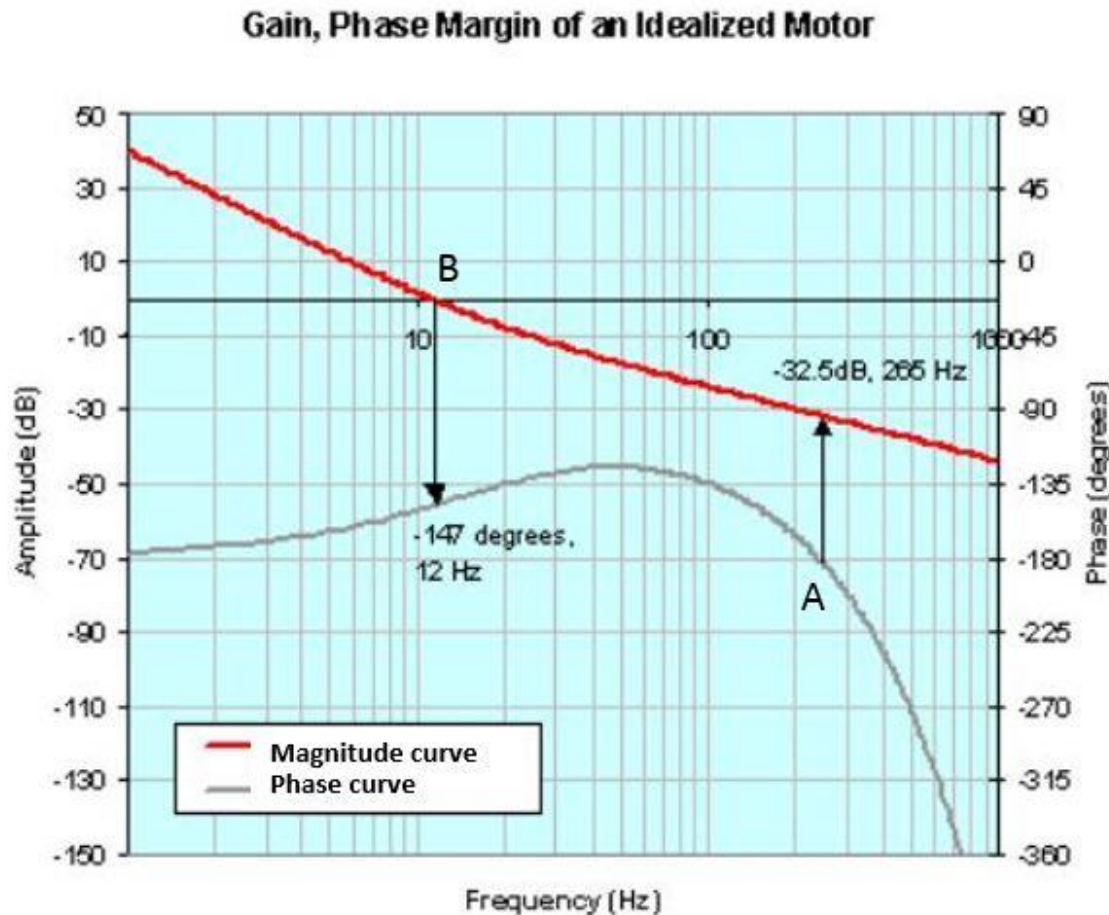
$$50/(s^3 + 9s^2 + 30s + 40)$$



(w = ω)

Example:

- A occurs at point where phase = -180°
- B occurs at point where magnitude = 0 dB



GM ≈ 32.5 dB

PM $\approx 33^\circ$

$(-147^\circ - (-180^\circ)) = 33^\circ$

Special Cases:

$G(s)H(s)$ has poles and/or zeros on the imaginary axis

Example:

$$G(s)H(s) = \frac{K}{s}$$

It has one pole at 0 (on the imaginary axis)

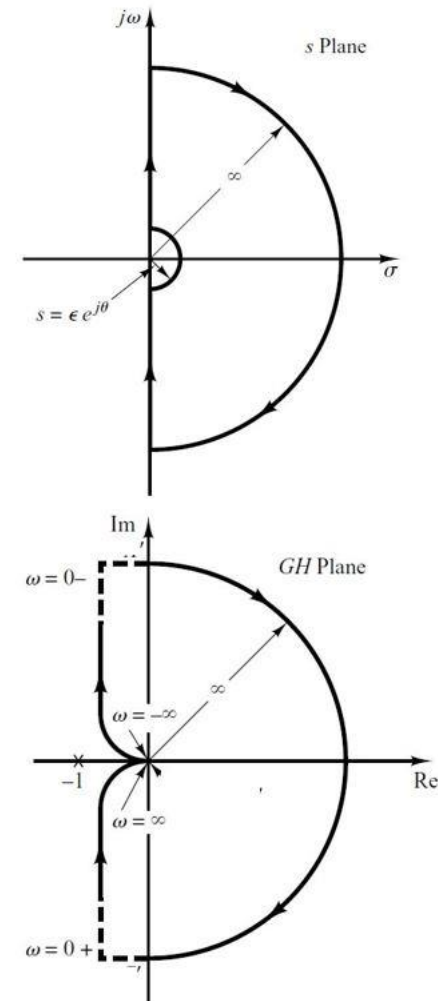
When $s = 0$, let the pole be $s = \varepsilon e^{j\theta}$ with $0 < \varepsilon \ll 1$ and $-90^\circ < \theta < +90^\circ$. Then

$$G(\varepsilon e^{j\theta})H(\varepsilon e^{j\theta}) = \frac{K}{\varepsilon e^{j\theta}} = \frac{K}{\varepsilon} e^{-j\theta}$$

As $\varepsilon: 0^- \rightarrow 0^+$, one has $\frac{K}{\varepsilon}: -\infty \rightarrow +\infty$, so that $G(j0^-)H(j0^-) = -\infty = +\infty = G(j0^+)H(j0^+)$

The $G(s)H(s)$ locus does not encircle $-1 + j0$

→ The closed-loop system is stable



Root Locus Design

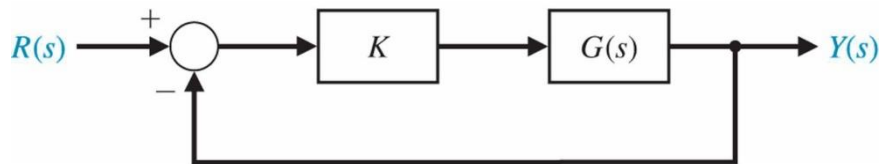
Using

MatLab: Control System Designer

Control System Toolbox™

[Free 30-Day Trial](#)

Example:



$$\frac{Y(s)}{R(s)} = \frac{s + 7}{s(s + 5)(s + 15)(s + 20)}$$

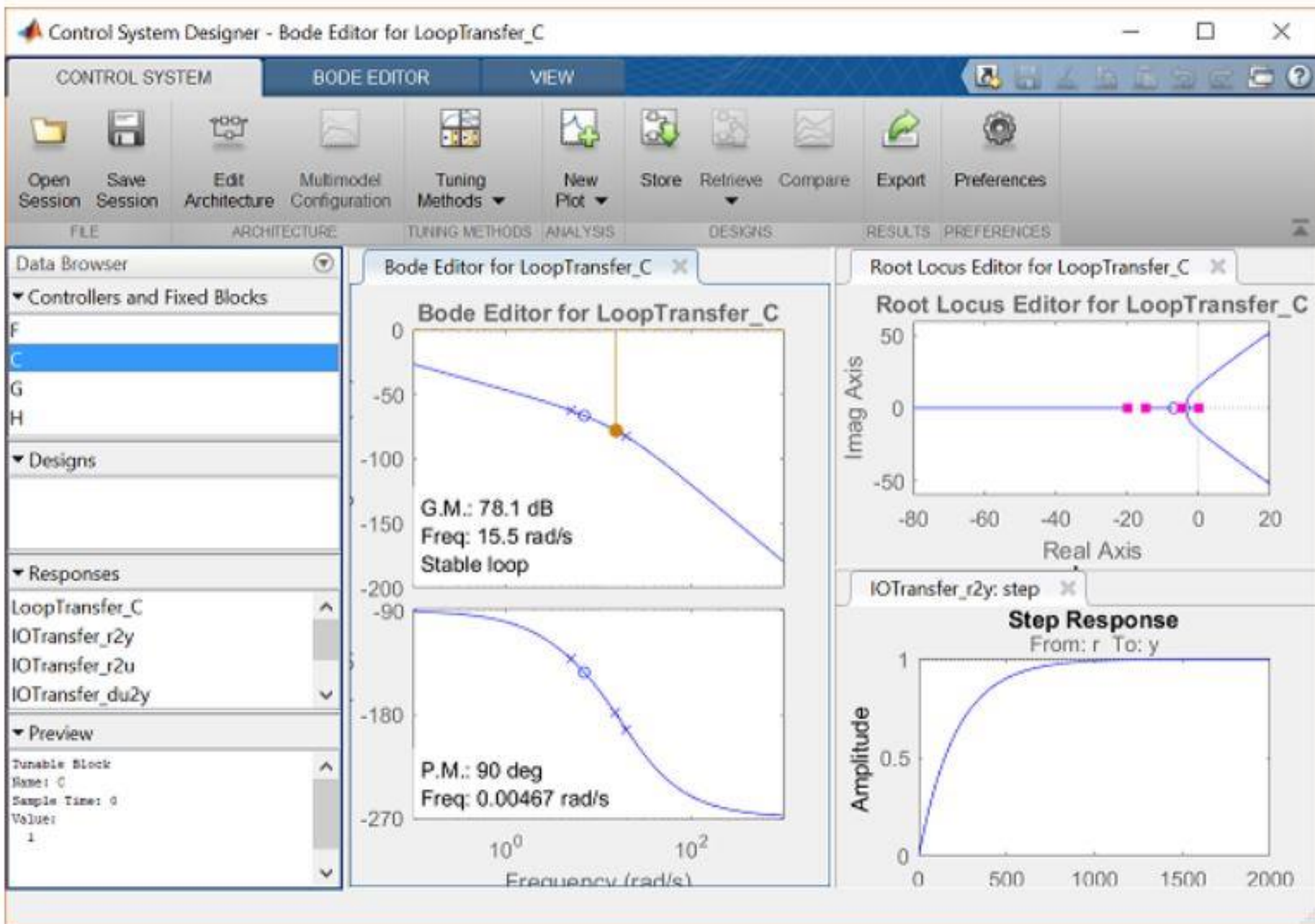
%MatLab code

```
s = tf('s');
```

```
plant = (s + 7)/(s*(s + 5)*(s + 15)*(s + 20));
```

```
controlSystemDesigner(plant)
```

← Command from Toolbox

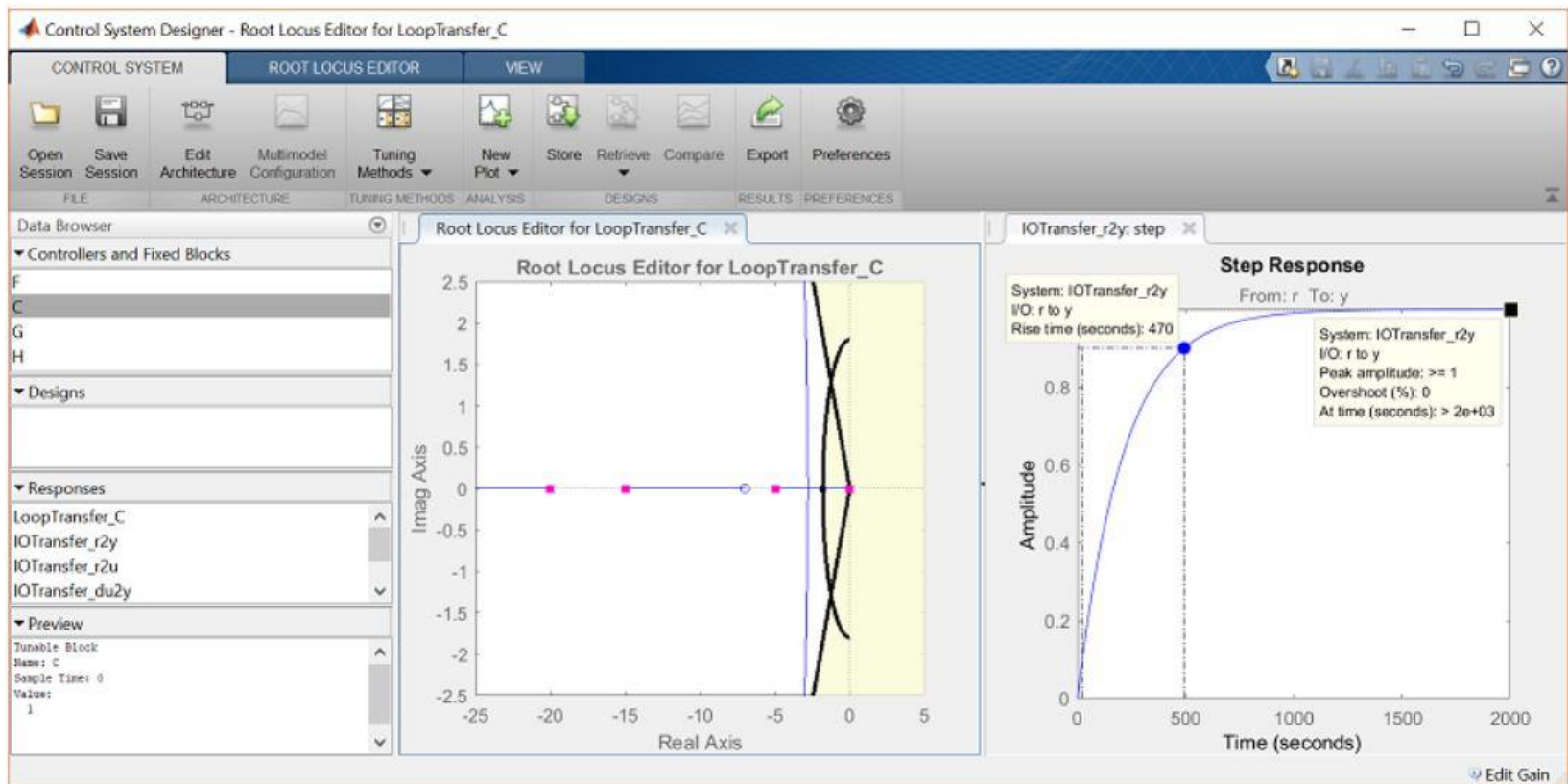


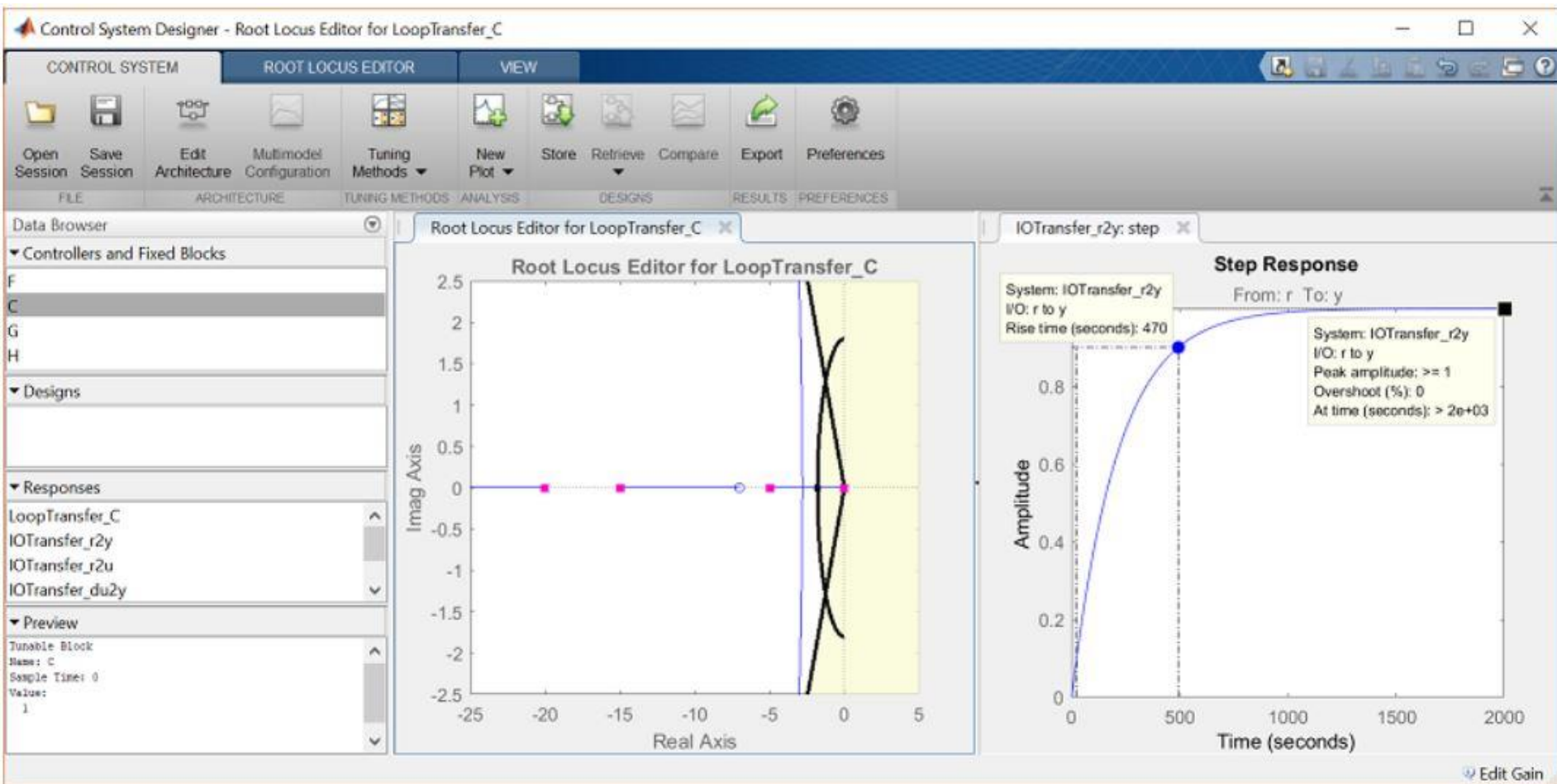
With default gain $K = 1$

Now, add the design requirements to the Root Locus plot, by right-clicking and selecting Design Requirements

Design Requirements: Settling Time, Percent Overshoot, Damping Ratio, Natural Frequency, Allowable Region

Example: As before, requiring $\zeta = 0.7$ and $\omega_n = 1.8$





On the left plot, the white area is **Acceptable Region** for the closed-loop poles

As the characteristics show on the Step Response, the **Overshoot** is acceptable but the **Rise Time** is much too large (which needs to be tuned, see next page)

Zoom into the **Root Locus** by right-clicking on an axis and selecting **Properties** followed by the label Limits

Change the real-axis limits: from -25 to 5

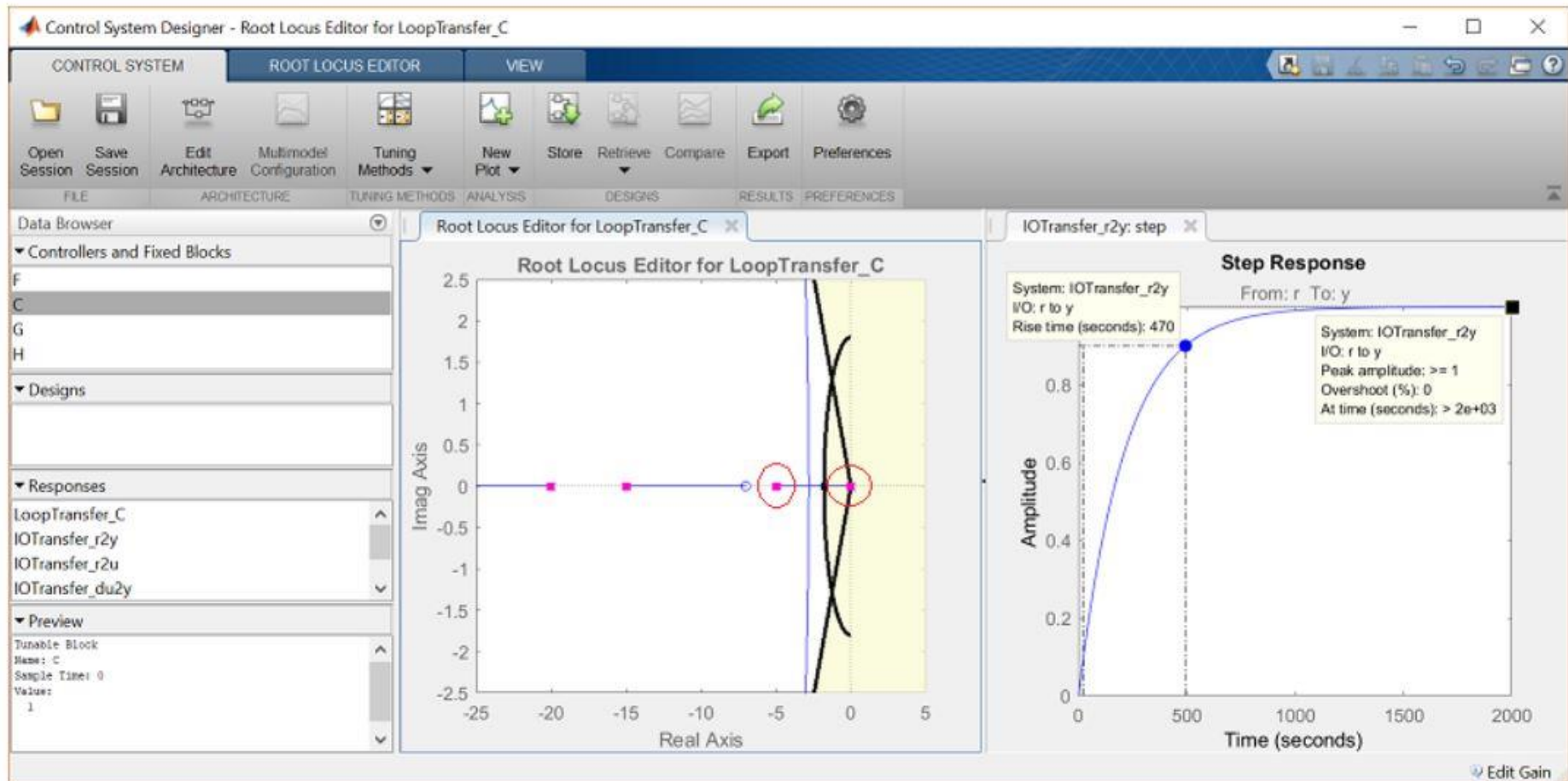
Change the imaginary axis limits: from -2.5 to 2.5

On the **Step Response** plot, right-click on the plot and go to **Characteristics** and select **Peak Response**. Then, select the characteristic **Rise Time**. Now, there should be two large dots on the screen indicating the values of these parameters. Click each of these dots to bring up a box with new information.

Both plots should appear as shown below, once the **Bode Plot** is closed:

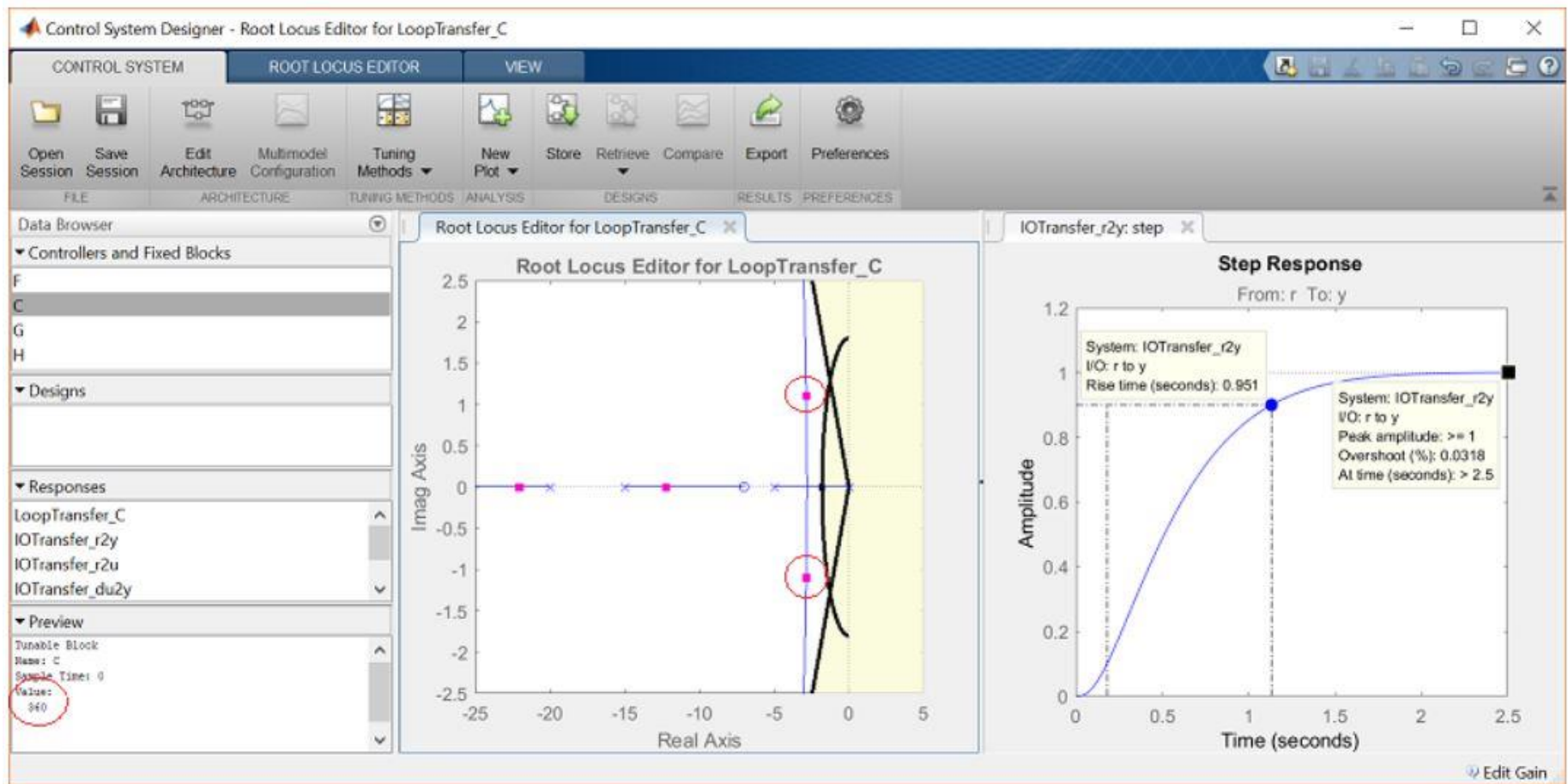
The pink boxes on the root locus show the corresponding closed-loop pole locations for the currently chosen control gain K

To fix it, choose a new value for the gain K , which can be changed directly on the root locus plot -- Click and drag the pink box closest to the imaginary axis (at the origin) to the acceptable region of the root locus plot as shown below:



In the **Preview** box of the window, it can be seen that the loop gain has been changed to 360

Inspecting the **Step Response** plot: Now, both requirements are met



END

