
EE3121

Differential Equations for Engineers

Tutorial 8a



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Tutorial

Problem 1:

Solve by the eigenvalue-eigenvector method the following differential equations:

(i)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

(ii)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

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Solution:

(i) Step 1: Find the eigenvalues of state matrix A , $\det(A - rI) = 0$

$$\Rightarrow \det \left(\begin{bmatrix} 1 - r & -1 \\ -5 & -3 - r \end{bmatrix} \right) = 0$$

we have

$$\begin{cases} r_1 = -4 \\ r_2 = 2 \end{cases}$$

Step 2: Find the eigenvectors of A

$$(A - r_1 I) \xi_1 = 0 \Rightarrow \xi_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

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$$(A - r_2 I) \xi_2 = 0 \Rightarrow \xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x(t) = c_1 \xi_1 e^{-4t} + c_2 \xi_2 e^{-2t}$$

then,

$$\Rightarrow x(t) = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

From the initial condition $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$,

$$\Rightarrow \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ 5c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

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$$\Rightarrow \begin{cases} c_1 = \frac{1}{3} \\ c_2 = \frac{11}{3} \end{cases}$$

$$\Rightarrow x(t) = \frac{1}{3} \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{-4t} + \frac{11}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

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(ii) Step 1: Find eigenvalues of state matrix $A = \begin{bmatrix} -6 & 4 \\ -5 & 2 \end{bmatrix}$.

$$\det(A - rI) = 0 \Rightarrow \det \left(\begin{bmatrix} -6 - r & 4 \\ -5 & 2 - r \end{bmatrix} \right) = 0$$

then, we can calculate

$$\begin{cases} r_1 = -2 + j2 \\ r_2 = -2 - j2 \end{cases}$$

Step 2: Find eigenvectors of A

$$(A - r_1 I) \xi_1 = 0 \Rightarrow \xi_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} j = a + bj$$

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The two real-valued fundamental solutions associated with complex conjugate eigenvalues $r_{1,2} = -\lambda \pm j\omega$ are

$$\begin{cases} x_1(t) = e^{-\lambda t} (a \cos \omega t - b \sin \omega t) \\ x_2(t) = e^{-\lambda t} (a \sin \omega t + b \cos \omega t) \end{cases}$$

therefore,

$$\Rightarrow \begin{cases} x_1(t) = e^{-2t} (a \cos 2t - b \sin 2t) = e^{-2t} \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} \cos 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 2t \right) \\ x_2(t) = e^{-2t} (a \sin 2t + b \cos 2t) = e^{-2t} \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} \sin 2t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos 2t \right) \end{cases}$$

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

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From the initial condition $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\Rightarrow \begin{cases} 2c_1 = -2 \\ 2c_1 + c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = -1 \\ c_2 = 3 \end{cases}$$

$$\begin{aligned} \Rightarrow x(t) &= -e^{-2t} \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} \cos 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 2t \right) \\ &\quad + 3e^{-2t} \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} \sin 2t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos 2t \right) \\ &= e^{-2t} \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cos 2t + \begin{bmatrix} 6 \\ 7 \end{bmatrix} \sin 2t \right) \end{aligned}$$

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Problem 2:

Solve by the matrix kernel method

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

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Solution:

Step 1: Find the eigenvalues of state matrix A , $\det(A - rI) = 0$

$$\det \left(\begin{bmatrix} 3 - r & 2 \\ 7 & -2 - r \end{bmatrix} \right) = 0$$

$$\Rightarrow \begin{cases} r_1 = -4 \\ r_2 = 5 \end{cases}$$

Step 2: Find eigenvectors of A

$$\begin{cases} (A - r_1 I) \xi_1 = 0 \Rightarrow \xi_1 = \begin{bmatrix} -2 \\ 7 \end{bmatrix} \\ (A - r_2 I) \xi_2 = 0 \Rightarrow \xi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$$

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Step 3: Find the fundamental matrix $\psi(t)$

$$\psi(t) = \begin{bmatrix} e^{r_1 t} \xi_1 & e^{r_2 t} \xi_2 \end{bmatrix} = \begin{bmatrix} -2e^{-4t} & e^{5t} \\ 7e^{-4t} & e^{5t} \end{bmatrix}$$

we can have

$$\psi(0) = \begin{bmatrix} -2 & 1 \\ 7 & 1 \end{bmatrix}, \psi^{-1}(0) = -\frac{1}{9} \begin{bmatrix} 1 & -1 \\ -7 & -2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -1 & 1 \\ 7 & 2 \end{bmatrix}$$

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Step 4: Find the state transition matrix $\phi(t)$

$$\begin{aligned}\phi(t) = \psi(t)\psi^{-1}(0) &= \begin{bmatrix} -2e^{-4t} & e^{5t} \\ 7e^{-4t} & e \end{bmatrix} \frac{1}{9} \begin{bmatrix} -1 & 1 \\ 7 & 2 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 2e^{-4t} + 7e^{5t} & -2e^{-4t} + 2e^{5t} \\ -7e^{-4t} + 7e^{5t} & 7e^{-4t} + 2e^{5t} \end{bmatrix}\end{aligned}$$

According to theorem,

$$x(t) = \phi(t, t_0)x(t_0) + \int_0^t \phi(t, \tau)B(\tau)u(\tau)d\tau$$

therefore,

$$x(t) = \phi(t)x(0) + \int_0^t \phi(\tau) \begin{bmatrix} 2 \\ -1 \end{bmatrix} d\tau = \int_0^t \phi(\tau) \begin{bmatrix} 2 \\ -1 \end{bmatrix} d\tau$$

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