

Due date: Saturday, 7 October 2023 (Week 5) before 11:59 p.m.

- Answer **all** questions. Show **all** steps.
- Please submit the softcopy of your work (as a **single pdf file**) to Canvas on or before the due date.

Questions

1. Find the value of a such that $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} -1 & 5 \\ 1 & a \end{pmatrix}$.
2. Given the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.
 - (a) Find the eigenvalues and the corresponding eigenvectors of A .
 - (b) Is A diagonalizable? If yes, express A in the form PDP^{-1} , where D is a diagonal matrix. State clearly the matrices P , D and P^{-1} in your answer.
 - (c) Hence, compute A^n , where n is a positive integer.
3. Given that $A = \begin{pmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 4 & 2 \end{pmatrix}$, whose characteristic polynomial is $(\lambda - 1)^3(\lambda + 2)$.
 - (a) Find eigenvalues and the corresponding eigenvectors of A .
 - (b) Is A diagonalizable? Give reasons.
4. Given the matrix $A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ and I is a 3×3 identity matrix.
 - (a) Find the eigenvalues of matrix A . Then for each eigenvalue, find a corresponding normalized eigenvector.
 - (b) Construct a diagonal matrix D and an invertible matrix P such that $AP = PD$.
 - (c) Verify $P^T P = I$ and find P^{-1} .
 - (d) Using your answers in parts (a) to (c), deduce the eigenvalues and normalized eigenvectors of matrix C , where $C = 4A^3 - A^2 + 2I$. (Hint: Do not compute the matrix C .)
5. Determine whether each of the following quadratic forms in the given number of variables is positive definite, negative definite, positive semidefinite, negative semidefinite or indefinite.
 - (a) $-4x_1^2 + 20x_1x_2 - 25x_2^2$ (in two variables)
 - (b) $4x_1^2 + x_2^2 - 6x_1x_3 + 2x_2x_3$ (in three variables)
 - (c) $3x_1^2 + 2x_2^2 + 4x_3^2 - 2x_1x_2 + 2x_1x_3 - 4x_2x_3$ (in three variables)

6. Consider the following quadratic form in three variables

$$q = ax_1^2 + ax_2^2 + ax_3^2 + 2x_1x_2 + 4x_2x_3 ,$$

where a is a real number.

By considering the principal minors of the associated symmetric matrix, determine all possible values of a for which the quadratic form q is

- (a) positive definite;
- (b) positive semidefinite;
- (c) negative definite;
- (d) negative semidefinite;
- (e) indefinite.

7. **Discovery Question.**

It is given that M is a 3×3 real symmetric matrix with eigenvalues $\lambda = 1, 1, -2$.

For $\lambda = 1$, two linearly independent eigenvectors are $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

For $\lambda = -2$, one of its corresponding eigenvectors is $\vec{v}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

Note that $\vec{v}_3$ is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$. However, the eigenvectors $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and

$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ for $\lambda = 1$ are not orthogonal. We can use the **Gram-Schmidt process** to find two orthogonal eigenvectors for $\lambda = 1$.

Read https://en.wikipedia.org/wiki/Gram-Schmidt_process to learn about the Gram-Schmidt process and then answer the following questions:

- (a) Use the Gram-Schmidt process to find two orthogonal eigenvectors of M corresponding to $\lambda = 1$.
- (b) Construct an orthogonal matrix P whose columns are orthonormal eigenvectors of M such that $P^T P = I$.

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