

**Problem 1 (Method of substitution)**

Compute the following integral using method of substitution:

- |                                                |                                                   |
|------------------------------------------------|---------------------------------------------------|
| (a) $\int \frac{e^{1+\frac{1}{x^2}}}{x^3} dx$  | (b) $\int x^2 \sec(1 - 2x^3) dx$                  |
| (c) $\int x^{11} \sqrt{1 + x^4} dx$            | (d) $\int x \cos^2(x^2) dx$                       |
| (e) $\int \sin 2x \sqrt{\cos x} dx$            | (f) $\int \frac{e^{2x}}{(1 + e^x)^3} dx$          |
| (g) $\int_1^2 x e^{x^2-1} dx$                  | (h) $\int_1^5 \frac{\sin^2(\ln x)}{x} dx$         |
| (i) $\int \frac{3x+2}{x^2+4} dx$               | (j) $\int \frac{2x+1}{x^2-2x+5} dx$               |
| (k) $\int \frac{4x}{3x^2+6x+19} dx$            | (l) $\int \frac{1}{x^2\sqrt{1-x^2}} dx$           |
| (m) $\int \frac{1}{(1+3x^2)^{\frac{3}{2}}} dx$ | (n) $\int \frac{3x}{\sqrt{4x^2+1}} dx$            |
| (o) $\int \sqrt{9-16x^2} dx$                   | (p) $\int \frac{1}{(x^2+6x+10)^{\frac{3}{2}}} dx$ |
| (q) $\int \sin^7 x dx$                         | (r) $\int \sin^3 x \cos^5 x dx$                   |

**Problem 2 (Integration by parts)**

Compute the following integrals using integration by parts:

- |                                                  |                                  |
|--------------------------------------------------|----------------------------------|
| (a) $\int x e^{-3x} dx$                          | (b) $\int_1^e \sqrt{x} \ln x dx$ |
| (c) $\int x^2 \sin x dx$                         | (d) $\int x \sin^2 x dx$         |
| (e) $\int x^2 \cos^{-1} x dx$                    | (f) $\int \tan^{-1} x dx$        |
| (g) $\int \csc^3 x dx$                           | (h) $\int \cos^3 x dx$           |
| (i) $\int_1^e \left(\frac{\ln x}{x}\right)^2 dx$ | (j) $\int e^x \sin 3x dx$        |

**Problem 3**

Compute the following integrals using suitable method. You may need to use method of substitution or integration by parts or both.

- |                                          |                                        |
|------------------------------------------|----------------------------------------|
| (a) $\int e^{2x} \sin(2e^x + 1) dx$      | (b) $\int_0^1 \sin(2\sqrt{x}) dx$      |
| (c) $\int_0^1 \ln(1 + \sqrt[3]{x}) dx$   | (d) $\int \cos(\ln x) dx$              |
| (e) $\int \sin 2x \ln(\sin x) dx$        | (f) $\int (x+1) \ln(x+3) dx$           |
| (g) $\int_1^2 \frac{e^{2x}}{e^x - 1} dx$ | (h) $\int x^3 \cos(3x^2) \sin(x^2) dx$ |
| (i) $\int x^2 \sqrt{4-x^2} dx$           | (j) $\int x^3 \sin(4+x^2) dx$          |

**Problem 4**

(a) Compute the integrals

$$\int e^{2x} \sin 3x \, dx \quad \text{and} \quad \int e^{2x} \cos 3x \, dx.$$

(b) Hence, compute the integrals

$$\int x e^{2x} \cos 3x \, dx.$$

(Hint: You need to eliminate  $x$  in the integrand so that you can compute the integral using the result of (a). Which technique should you use: Method of substitution and/or integration by parts?)

**Problem 5**

Let  $f(x)$  be a differentiable function on  $[a, b]$  such that  $\int_a^b f(x) dx = 0$  and  $f(a) = f(b) = 1$ . Find the value of  $\int_a^b x f'(x) dx$ .

**Problem 6**

Let  $f(x)$  be the continuous function, show that for any  $a > 0$ , we must have

$$\int_0^a x^3 f(x^2) dx - \frac{1}{2} \int_0^{a^2} x f(x) dx = 0.$$

**Problem 7**

Let  $f(x)$  be twice differentiable function on  $[0, 1]$  such that  $f(0) = f(1) = 1$  and  $\int_0^1 f(x) dx = 1$ .

Using integration by parts, find the value of

$$\int_0^1 x(1-x)f''(x) dx.$$

(Hint: The technique in Problem 5 may be useful.)

**Problem 8**

For non-negative integer  $n$ , we define the integral

$$I_n = \int_0^1 x^n e^{-3x} dx.$$

(a) Deduce the following reduction formula for  $I_n$ :

$$I_n = -\frac{1}{3} e^{-3} + \frac{n}{3} I_{n-1}, \quad n \geq 1.$$

(b) Using the reduction formula in (a), find the value of

$$\int_0^1 x^3 e^{-3x} dx.$$

**Problem 9**

For any real number  $a \neq -1$  and non-negative integer  $n$ , we define the integral as

$$I_n = \int_1^e x^a (\ln x)^n dx.$$

(a) Deduce the following reduction formula for  $I_n$ :

$$I_n = \frac{e^{a+1}}{a+1} - \frac{n}{a+1} I_{n-1}, \quad n \geq 1.$$

(b) Using the reduction formula in (a), find the value of

$$\int_1^e x^2 (\ln x)^3 dx.$$

(c) What is the value of  $I_n$  when  $a = -1$ ?

**Problem 10**

For non-negative integer  $n$ , we define the integral

$$I_n = \int (x^2 + a^2)^n dx.$$

(a) Show that

$$I_n = \frac{1}{2n+1} x(x^2 + a^2)^n + \frac{2n}{2n+1} a^2 I_{n-1}, \quad n \geq 1.$$

(b) Compute the integral

$$\int (x^2 + a^2)^4 dx.$$

**Problem 11**

For any non-negative integer  $n$ , we define the integral  $I_n$  as

$$I_n = \int x^n \cos 3x \, dx.$$

(a) Deduce the following reduction formula for  $I_n$ :

$$I_n = \frac{1}{3} x^n \sin 3x + \frac{n}{9} x^{n-1} \cos 3x - \frac{n(n-1)}{9} I_{n-2}, \quad n \geq 2.$$

(Hint: You need to use integration by parts twice)

(b) Using the result of (a), compute the integral

$$\int x^4 \cos 3x \, dx.$$

**Problem 12**

Consider the integral

$$I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n x \, dx.$$

(a) Show that

$$I_n = \frac{n-1}{n} I_{n-2}, \quad n \geq 2.$$

(b) Using the reduction formula obtained in (a), find the value of

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, dx, \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx.$$

(c) Find the value of

$$\int_{-1}^1 (1-x^2)^{\frac{5}{2}} dx.$$

**Problem 13**

Consider the integral

$$I_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx.$$

(a) Deduce the following reduction formula for  $I_n$ :

$$I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}, \quad n \geq 2.$$

(b) Using the reduction formula in (a), find the value of

$$\int_0^1 (1+x^2)^{\frac{3}{2}} dx \quad \text{and} \quad \int_2^{2\sqrt{2}} x^2 \sqrt{x^2-4} \, dx.$$

**Problem 14 (A bit harder)**

For any non-negative integer  $n$ , we define the integral

$$I_n = \int_0^a x^n \sqrt{a^2 - x^2} dx, \quad a > 0.$$

(a) Deduce the reduction formula for  $I_n$ :

$$I_n = \left( \frac{n-1}{n+2} \right) a^2 I_{n-2}, \quad n \geq 2.$$

(b) Compute the integral

$$\int_0^a x^2 (a^2 - x^2)^{\frac{3}{2}} dx.$$

(Hint:  $(a^2 - x^2)^{\frac{3}{2}} = (a^2 - x^2) \sqrt{a^2 - x^2}$ .)

**Problem 15 (Harder)**

For any non-negative integer  $n$ , we define the integral

$$I_n = \int (\sin^{-1} x)^n dx.$$

(a) Show that

$$I_n = x(\sin^{-1} x)^n + n(\sin^{-1} x)^{n-1} \sqrt{1-x^2} - n(n-1)I_{n-2}, \quad n \geq 2.$$

(b) Hence, compute the integrals

$$\int (\sin^{-1} x)^3 dx, \quad \int \frac{x(\sin^{-1} x)^5}{\sqrt{1-x^2}} dx.$$

(Hint: You need to apply integration by parts on the second integral.)

**Problem 16 (Method of Partial Fractions)**

Compute the following integrals using method of partial fraction.

(☺Hint: In some cases (say, (a),(c),(f),(h)), you may need to factorize the denominator)

(a)  $\int \frac{1}{3x - x^2} dx$

(b)  $\int \frac{2x^2 - 5x + 5}{(x-1)^2(x-2)} dx$

(c)  $\int \frac{3x^4 - 5x^3 + x^2 + 2x + 1}{3x^3 - 2x^2 - x} dx$

(d)  $\int \frac{3x^2 + 10}{(x+3)(x^2 - 6x + 10)} dx$

(e)  $\int \frac{-7x + 19}{(x^2 - 4x + 9)(2x + 1)} dx$

(f)  $\int \frac{8x^2 - 3x - 2}{4x^3 - 3x + 1} dx$

(g)  $\int \frac{x^2 - 5x - 5}{(x-2)(x^2 + 2x + 3)} dx$

(h)  $\int \frac{x^5 + 2x^4 - x + 2}{x^3 + 2x^2 - x - 2} dx$

(i)  $\int \frac{2x^2 - x + 1}{x^3(x-1)} dx$

(j)  $\int \frac{3x^3 - 2x - 20}{(x^2 + 3)(2x^2 - 6x + 5)} dx$

\*(k)  $\int \frac{6x^3 - 27x^2 + 5x - 1}{(x-2)^2(4x^2 + 1)} dx$

\*(l)  $\int \frac{x^2 + 2x + 4}{x(x^2 + 2)^2} dx$