

Part A: Basic Concept**Problem 1**

Let $A = (1,1,0)$, $B = (0,2,3)$ and $C = (2,-1,0)$ be three points in a plane.

- (a) Write down the position vectors of A , B and C .
- (b) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{CA}$.
- (c) Is $\overrightarrow{AB} = \overrightarrow{BC}$? Explain your answer.
- (d) Find the unit vector of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.
- (e) (i) Let $\vec{a}$ be a vector with magnitude 3 and the direction is opposite to that of $\overrightarrow{AB}$. Find the vector $\vec{a}$.
(ii) Let $\vec{b}$ be a vector with magnitude 5 and the direction is opposite to that of $\overrightarrow{BC}$. Find the vector $\vec{b}$.

Problem 2

Let $A = (0,1,-1)$ and $B = (1,2,0)$ be two points in a plane. Let X be a point between A and B such that $AX:XB = 2:1$.

- (a) Find $\overrightarrow{AB}$ and $\overrightarrow{AX}$.
- (b) Hence, find the coordinate of X by finding its position vector $\overrightarrow{OX}$. (Hint: $\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA}$).

Problem 3

Let $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j}$ be two vectors.

- (a) Find $|\vec{a}|$ and $|\vec{a} - 2\vec{b}|$.
- (b) Find the unit vector of $\vec{b}$.
- (c) Let $\vec{c}$ be another vector with magnitude $|2\vec{a} + \vec{b}|$ and its direction is same as that of $\vec{b}$. Find the vector $\vec{c}$.

Part B: Scalar Product and its application**Problem 4**

Let $\vec{a} = \vec{i} + 3\vec{j} - 2\vec{k}$ and $\vec{b} = -2\vec{i} + \vec{j} + 3\vec{k}$ be two vectors.

- (a) Find $\vec{a} \cdot \vec{b}$.
- (b) Find the angle between the vectors $\vec{a}$ and $\vec{b}$.
- (c) Let $\vec{c} = 3\vec{i} + x\vec{j} - 2\vec{k}$ be a vector which is perpendicular to $\vec{b}$, find the value of x .
- (d) Let $\vec{d} = y\vec{a} + 3\vec{b}$ be a vector which is perpendicular to $\vec{a} - \vec{b}$, find the value of y .

Problem 5

- (a) Let $A = (1,1,0)$, $B = (0,1,2)$ and $C = (2,1,0)$ be three points in a plane, find $\angle ABC$.
- (b) It is given that $D = (x, 1, 3)$, $E = (1, 2, 3)$ and $F = (4, -4, 1)$ are three points in a plane. Suppose that DE is perpendicular to EF , find the value of x .

Problem 6

Let $A = (4,2)$, $B = (1,1)$ and $C = (2,3)$ be three points in a 2D-plane. Let $D = (3,3)$ be another point in the same plane.

(a) Find $\angle ABC$.

(b) Is BD an angle bisector of $\angle ABC$? Explain your answer. (Hint: A clear figure may help)

Problem 7

Let A , B and C be three points in a plane such that $|\overrightarrow{AB}| = |\overrightarrow{AC}| = 4$ and $\overrightarrow{AB} \cdot \overrightarrow{AC} = 2$. Find the length of BC .

Problem 8

Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = 1$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 1$.

(a) Find the angle between the vectors $\vec{a}$ and $\vec{b}$.

(b) Find the value of $(3\vec{a} - 2\vec{b}) \cdot (\vec{a} + 3\vec{b})$ and $|\vec{a} - 2\vec{b}|$.

(c) Find the angle between two vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$.

Problem 9

Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = 2$ and $|\vec{b}| = 3$ and the angle between these two vectors is $\cos^{-1} \frac{3}{5}$.

(a) Are the vector $\vec{a} - 2\vec{b}$ and $-9\vec{a} + 2\vec{b}$ perpendicular to each other? Explain your answer.

(b) If the angle between the vectors $\vec{a}$ and $\vec{a} + k\vec{b}$ is 60° , find the value of k .

Problem 10

Find the projection vector of $\vec{a}$ onto $\vec{b}$ ($proj_{\vec{b}} \vec{a}$) for each of the following set of vectors $\vec{a}$ and $\vec{b}$.

(a) $\vec{a} = 3\vec{i} - 4\vec{j}$ and $\vec{b} = \vec{i} - 18\vec{j}$.

(b) $\vec{a} = 2\vec{i} - 3\vec{j} - 6\vec{k}$ and $\vec{b} = 6\vec{i} - 2\vec{j} + 11\vec{k}$.

(c) $\vec{a} = -\vec{i} + 2\vec{j} + 2\vec{k}$ and $\vec{b} = \vec{i} + 7\vec{j} + 7\vec{k}$.

Problem 11

(a) Let L_1 be a line passing through the points $A = (1,1,0)$ and $B = (-1,2,3)$, find the shortest distance between a point $C = (0,1,0)$ and the line L_1 .

(b) Let L_2 be a line passing through the points $D = (2,-1,1)$ and $E = (0,0,1)$, find the shortest distance between a point $F = (1,3,-1)$ and the line L_2 .

Part C: Vector Product and Scalar Triple Product

Problem 12

Find the value of $\vec{a} \times \vec{b}$ for each of following set of the vectors $\vec{a}$ and $\vec{b}$.

(a) $\vec{a} = \vec{i} + 3\vec{j}$ and $\vec{b} = -2\vec{j} + 5\vec{k}$.

(b) $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$ and $\vec{b} = -3\vec{i} + 2\vec{j} + 5\vec{k}$

(c) $\vec{a} = -3\vec{i} + \vec{j} + 3\vec{k}$ and $\vec{b} = 6\vec{j} + \vec{k}$

(d) $\vec{a} = \vec{j} + \vec{k}$ and $\vec{b} = 3\vec{i} - \vec{j} + 2\vec{k}$.

Problem 13

Let $\vec{a}$ and $\vec{b}$ be two vectors in a plane, what is the value of $\vec{a} \cdot (\vec{a} \times \vec{b})$? (Hint: Think about the relationship between the vector $\vec{a}$ and $\vec{a} \times \vec{b}$.)

Problem 14

Let $A = (1, 2, 0)$, $B = (3, -1, -2)$ and $C = (-2, 0, 1)$ be three points in the plane.

- Find a vector which is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
- Let $\vec{a}$ be a vector with same magnitude as that of $\overrightarrow{BC}$ and it is perpendicular to both vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Find the vector $\vec{a}$.
- (A bit harder) Find the equation of the plane containing the points A , B and C . (Hint: See the remark of Example 12 of Lecture Note 4.)

Problem 15

Let $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = 4\vec{i} - 4\vec{j} + 3\vec{k}$ be two vectors.

- Find a vector $\vec{c}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$.
- Find the area of the triangle with $\vec{a}$ and $\vec{b}$ as its adjacent sides.
- Find the equation of the plane passing through a point $(1, 1, 1)$ and containing the vectors $\vec{a}$ and $\vec{b}$. (Hint: See the remark of Example 12 of Lecture Note 4.)
- Let $\vec{d} = \vec{i} + 2\vec{k}$ be a vector. Determine if the vectors $\vec{a}$, $\vec{b}$ and $\vec{d}$ are coplanar by finding the volume of parallelepiped with $\vec{a}$, $\vec{b}$ and $\vec{d}$ as adjacent sides.

Problem 16

Let $A = (3, -1, 3)$, $B = (0, 7, -2)$ and $C = (-9, 3, -3)$ be three points in a plane. Find the area of the triangle ABC . Also find the area of the parallelogram with AB and AC as the adjacent sides.

Problem 17

In each of the following, determine whether the given three points are collinear.

- $A = (-1, 0, 1)$, $B = (2, 4, 1)$ and $C = (1, 1, 0)$
- $A = (1, 2, -1)$, $B = (-1, 1, 2)$ and $C = (3, 3, -4)$

Problem 18

In each of the following, find the volume of parallelepiped with the given four points as the adjacent vertices. Hence determine if the given four points are coplanar.

- $A = (2, 1, -1)$, $B = (0, 1, 1)$, $C = (-2, -1, 5)$ and $D = (2, 3, -3)$.
- $A = (1, 1, 1)$, $B = (1, -1, 3)$, $C = (-1, 0, 2)$ and $D = (2, -1, 2)$.

Problem 19

- Let π_1 be a plane containing the points $A = (3, -2, 0)$, $B = (2, 0, 3)$ and $C = (1, -1, 1)$, find the shortest distance between point $D = (1, 0, -1)$ and the plane π_1 .
- Let π_2 be a plane passing through a point $A = (2, 1, -6)$. It is also given that the vector $\vec{n} = -\vec{i} - \vec{j} - \vec{k}$ is perpendicular to the plane π . Find the shortest distance between $B = (1, -1, 1)$ and the plane π_2 .

Problem 20

- Let L_1 be a line passing through the points $(5, 0, -1)$ and $(6, 2, -2)$. We let L_2 be another line passing through the points $(2, 4, 0)$ and $(3, 3, 1)$. Find the shortest distance between the line L_1 and L_2 .

- (b) Let L_1 be a line passing through the points $(1,1,1)$ and $(2,1,2)$. We let L_2 be another line passing through the points $(2,1,0)$ and $(3,2,0)$. Find the shortest distance between the line L_1 and L_2 .

Part D: Linear Independence of vectors

Problem 21

Determine if each of the following set of vectors are linearly independent.

- (a) $\vec{a} = \vec{i} - 2\vec{j}$ and $\vec{b} = 2\vec{i} + \vec{j}$.
 (b) $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + 5\vec{j} + \vec{k}$ and $\vec{c} = 3\vec{i} + 2\vec{j} - 3\vec{k}$.
 (c) $\vec{a} = \vec{i} + 2\vec{j} - 5\vec{k}$, $\vec{b} = -\vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{c} = 3\vec{i} - 2\vec{j} + \vec{k}$.

Problem 22

Find the value of m such that the following sets of vectors are *linearly dependent*.

$$\vec{a} = (1 - m)\vec{i} + 6\vec{j} + 5\vec{k}, \quad \vec{b} = 2\vec{i} - m\vec{j}, \quad \vec{c} = -5m\vec{j} + 5\vec{k}.$$

Part E: A bit harder problems

Problem 23

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-zero vectors. Show that

- (a) $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$.
 (b) If $\vec{a}$ and $\vec{b}$ are perpendicular, then $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$.
 (c) If $\vec{a}$ and $\vec{b}$ are parallel, then the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are also parallel. (Hint: If two vectors are parallel, what is the angle between them? What can you say about the vector product of these two vectors?)
 (d) $\tan \theta = \frac{|\vec{a} \times \vec{b}|}{\vec{a} \cdot \vec{b}}$ where θ is the angle between $\vec{a}$ and $\vec{b}$.
 (e) $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$.