

MA1502 / GE1359 Algebra
Semester A, 2020-21
Test

Date: Monday, 2nd November 2020

Time allowed: 90 minutes (9:45 a.m. – 11:15 a.m.)

Submission time: 15 minutes (11:15 a.m. – 11:30 a.m.)

Instructions to candidates:

This paper has FOUR pages (including this cover page).

The formula sheet of trigonometric identities is provided on page 4.

This paper has SEVEN questions.

The questions will begin on page 2.

Answer ALL questions. Show ALL steps to get full mark.

Please submit your work as a single pdf file (with the signed Academic Honesty Pledge as the front page) to Canvas before 11:30 a.m. HK time on 2nd November 2020.

Question 1

For the following real-valued function, find its largest possible domain and largest possible range. Give your answers in set/interval notations.

$$f(x) = \frac{1}{9 - x^2}. \quad [15 \text{ marks}]$$

Question 2

Let g be a function defined by $g(x) = -x^2 + 4x - 1$ for $x \leq 1$.

- (a) Sketch the graph of $y = g(x)$ and show that $g(x)$ is a one-to-one function. [7 marks]
- (b) Find $g^{-1}(x)$ and the largest possible domain of $g^{-1}(x)$, where g^{-1} is the inverse function of g . [8 marks]

Question 3

- (a) Prove the identity: $\frac{1 - \cos x}{1 + \cos x} = \tan^2\left(\frac{x}{2}\right)$. [5 marks]
- (b) Given that $\sin A = -\frac{4}{5}$, where $180^\circ < A < 270^\circ$. Without using calculator, find the value of $\tan\left(\frac{A}{2}\right)$. (Hint: You may use (a).) [8 marks]

Question 4

- (a) Express $5 \sin(3x) + 12 \cos(3x)$ as the form $R \sin(3x + \theta)$, where $R > 0$ and $\theta \in (0, \pi/2)$ in radians. [9 marks]
- (b) Using your answer in (a), or otherwise, find the general solution (in radians) of the equation

$$5 \sin(3x) + 12 \cos(3x) = \frac{13}{2}. \quad [8 \text{ marks}]$$

Question 5

Express $\frac{-5x^2 + 6x + 23}{(x-1)^2(x^2+5)}$ into partial fractions. [12 marks]

Question 6

Using Gaussian elimination, determine the value(s) of p for which the system of linear equations

$$\begin{cases} x + 2y - z - 3w = 3 \\ 2x - y + 3z + 4w = 1 \\ -3x - y - 2z - w = p \end{cases}$$

is consistent and then solve the system in this case.

[13 marks]

Question 7

Let α, β be the roots of the quadratic equation $x^2 + qx + r = 0$, where q and r are non-zero real numbers.

(a) Find $\alpha^3 + \beta^3$ in terms of q and r .

[6 marks]

(b) Find a quadratic equation (in terms of q and r) whose roots are

$$\frac{\alpha}{\beta^2} + 2 \quad \text{and} \quad \frac{\beta}{\alpha^2} + 2.$$

[9 marks]

- End -

Trigonometric identities - Formula sheet

➤ Compound Angle Formulae

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

➤ Double Angle Formulae

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

➤ Half Angle Formulae

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

➤ Product-to-Sum Formulae

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

$$\sin A \sin B = -\frac{1}{2}[\cos(A + B) - \cos(A - B)]$$

➤ Sum-to-Product Formulae

$$\sin x + \sin y = 2 \sin\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right)$$

$$\sin x - \sin y = 2 \cos\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right)$$

$$\cos x + \cos y = 2 \cos\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right)$$

$$\cos x - \cos y = -2 \sin\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right)$$