

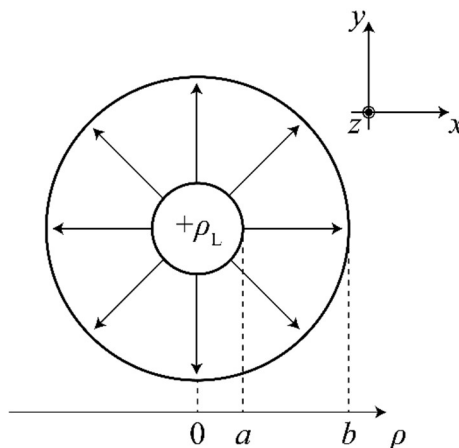
EE2104 Introduction to Electromagnetics
Assignment 4 Solution

Deadline (for uploading): 11:59 pm, 4 Dec 2021
Please put down your name and university ID.

1. Consider a long coaxial cable of inner and outer radii a and b , respectively. The gap is filled with material of relative permittivity ϵ_r and relative permeability μ_r . The length of the cable is l . Using the formulae for the fields associated with infinite line charge and current, find
- the capacitance C
 - the inductance L
 - The concept of characteristic impedance $Z = (L/C)^{1/2}$ will be introduced. For $\epsilon_r = 6$, $\mu_r = 1$, $b = 5$ mm, and $a = 1$ mm, evaluate Z and state its unit.

Solution:

Let the cable placed along the z -axis as shown by the figure below.



- (a) Assume that the inner conductor is charged with a charge density of $+\rho_L$ (C/m). Thus, we have

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\epsilon_r\rho} \hat{\mathbf{a}}_\rho$$

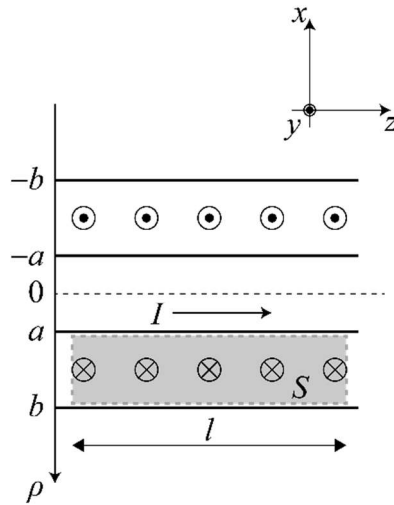
Voltage V in reference to the outer conductor can be calculated as

$$V = - \int_{\rho=b}^a \mathbf{E} \cdot d\mathbf{L} = - \int_{\rho=b}^a \frac{\rho_L}{2\pi\epsilon_0\epsilon_r\rho} d\rho = \frac{\rho_L}{2\pi\epsilon_0\epsilon_r} \ln\left(\frac{b}{a}\right)$$

So the capacitance is

$$C = \frac{\rho_L \cdot l}{V} = \frac{2\pi\epsilon_0\epsilon_r l}{\ln\left(\frac{b}{a}\right)}$$

(b) Assume that current $I\hat{\mathbf{a}}_z$ flows along the inner conductor.



We have

$$\mathbf{B} = \frac{\mu_0 \mu_r I}{2\pi \rho} \hat{\mathbf{a}}_\phi$$

Thus, flux through the surface S in the above figure can be expressed as

$$\Phi = \int_S \frac{\mu_0 \mu_r I}{2\pi \rho} dS = l \int_{\rho=a}^b \frac{\mu_0 \mu_r I}{2\pi \rho} d\rho = l \frac{\mu_0 \mu_r I}{2\pi} \ln\left(\frac{b}{a}\right)$$

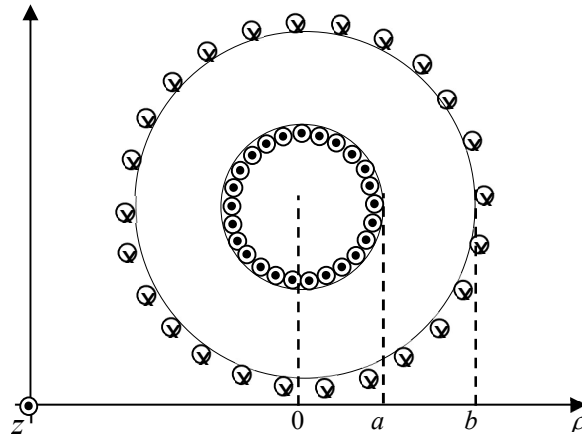
So the inductance is

$$L = \frac{\Phi}{I} = \frac{\mu_0 \mu_r}{2\pi} \ln\left(\frac{b}{a}\right) \cdot l$$

(c) The impedance is

$$Z = \sqrt{L/C} = \sqrt{\left[\frac{\mu_0 \mu_r}{2\pi} \ln\left(\frac{b}{a}\right) \cdot l \right] / \left[\frac{2\pi \epsilon_0 \epsilon_r l}{\ln\left(\frac{b}{a}\right)} \right]} = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}} \cdot \ln\left(\frac{b}{a}\right) \cdot \frac{1}{2\pi} = 39.4 \, (\Omega)$$

2. Consider a torus of $N = 25$ turns with inner and outer radii $a = 20$ mm and $b = 60$ mm, respectively. The torus is put at the origin with the z -axis as the axis of rotation. The direction of the current $I = 5$ A is such that the inner current flows towards the $+z$ -direction. In the cylindrical coordinates, the following figure shows a cross-section of the torus. Find the magnetic flux density $\mathbf{B}$ at $\rho = \frac{a+b}{2}$ in the middle of the torus.



Solution:

For the torus structure, we choose a circle centered at the origin on the plane $z = 0$ as the integration loop L . The radius ρ is treated as a variable.

Using Ampere's law,

$$\oint_L \mathbf{H} d\mathbf{L} = I$$

where L is integration in the counter-clockwise route and I is the current enclosed in the right-hand route.

By symmetry, we have

$$H(\rho, \phi, 0) = H(\rho) \hat{\mathbf{a}}_\phi$$

Thus,

$$\oint_L \mathbf{H} d\mathbf{L} = H(\rho) (\hat{\mathbf{a}}_\phi \cdot \hat{\mathbf{a}}_\phi) dL = I \Rightarrow H(\rho) 2\pi\rho = I \Rightarrow H(\rho) = \frac{I}{2\pi\rho}$$

At $\rho = \frac{a+b}{2}$, we have

$$H(\rho) = \frac{I}{2\pi\rho} = \frac{NI}{\pi(a+b)}$$

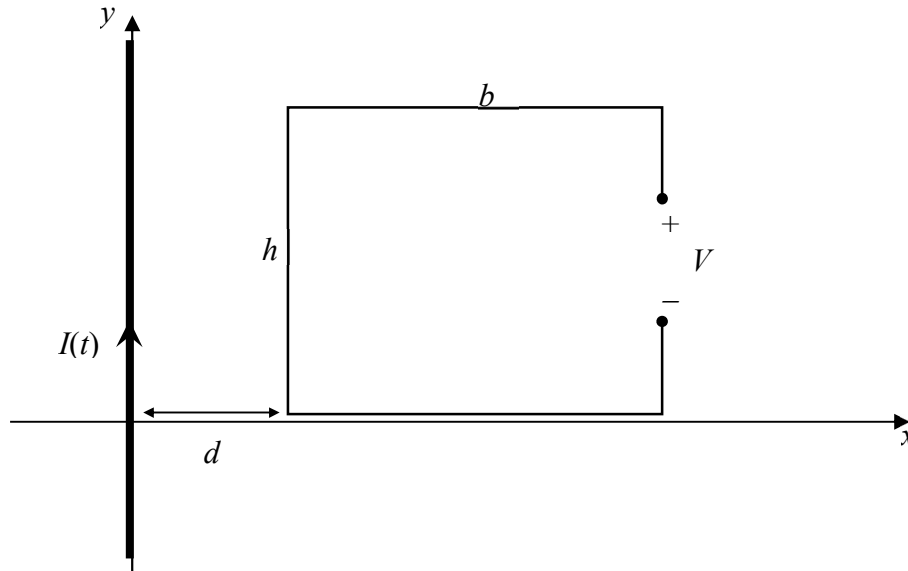
So

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 NI}{\pi(a+b)} \hat{\mathbf{a}}_\phi = 6.25 \times 10^{-4} \hat{\mathbf{a}}_\phi \text{ (T)}$$

3. An infinitely long wire of current I flowing towards the $+y$ -direction is put on the y -axis. The current is oscillating as:

$$I = I_0 \cos(2\pi f_0 t)$$

where I_0 is the amplitude and f_0 is the frequency. A rectangular open wire loop of height h and width b is put at distance d away, as the following figure shows. Find the induced voltage V across the loop.



Solution:

The magnetic field on the xy -plane is

$$\mathbf{B} = -\frac{\mu_0 I}{2\pi x} \hat{\mathbf{a}}_z = -\frac{\mu_0 I_0}{2\pi x} \cos(2\pi f_0 t) \hat{\mathbf{a}}_z$$

The flux contained by the coil is

$$\Phi = \int_{\text{coil area}} \mathbf{B} \cdot d\mathbf{S} = \int_{x=d}^{x=d+b} \frac{\mu_0 I_0}{2\pi x} \cos(2\pi f_0 t) (-\hat{\mathbf{a}}_z \cdot -\hat{\mathbf{a}}_z) h dx = \frac{h\mu_0 I_0}{2\pi} \cos(2\pi f_0 t) \ln \left(1 + \frac{b}{d}\right)$$

So the induced voltage is

$$V = -\frac{d\Phi}{dt} = f_0 h \mu_0 I_0 \ln \left(1 + \frac{b}{d}\right) \sin(2\pi f_0 t)$$

where clockwise direction is assumed as the direction of the induced current.

4. The following considers electromagnetic wave propagating in a dielectric of permittivity ϵ . The permeability remains as μ_0 . There are no free charges inside the dielectric.
- State the Maxwell's equations in terms of $\mathbf{E}$ and $\mathbf{H}$.
 - Derive the wave equation for $\mathbf{E}$.
 - An electromagnetic wave is described by:

$$\begin{aligned}\mathbf{E} &= E_0 \cos(kz - \omega t) \hat{\mathbf{a}}_x \\ \mathbf{H} &= H_0 \cos(kz - \omega t) \hat{\mathbf{a}}_y\end{aligned}$$

where the amplitudes E_0 and H_0 , the wavenumber k , and the angular frequency ω are constants. The propagation direction is denoted as the z -direction. Find H_0 in terms of E_0 .

Solution:

For electromagnetic waves propagating inside a dielectric, we have

$$\begin{aligned}\mathbf{D} &= \epsilon \mathbf{E} \\ \mathbf{B} &= \mu_0 \mathbf{H}\end{aligned}$$

- The Maxwell's equations are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times \mathbf{D} = 0$$

$$\nabla \times \mathbf{B} = 0$$

where ρ_v and $\mathbf{J}$ do not exist because of the absence of free-charges. Thus, we have

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \times \mathbf{H} = 0$$

- Using the curl equation, we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial}{\partial t} \nabla \times \mathbf{H} = -\mu_0 \epsilon \frac{\partial^2}{\partial t^2} \mathbf{E}$$

By $\nabla \times \nabla \times = \nabla \nabla \cdot - \nabla^2$, we have

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \epsilon \frac{\partial^2}{\partial t^2} \mathbf{E}$$

Since $\nabla \cdot \mathbf{E} = 0$, we obtain the wave equation:

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon \frac{\partial^2}{\partial t^2} \mathbf{E}$$

(c) From the Maxwell's equations,

$$\nabla \times [E_0 \cos(kz - \omega t) \hat{\mathbf{a}}_x] = -\mu_0 \frac{\partial}{\partial t} [H_0 \cos(kz - \omega t) \hat{\mathbf{a}}_y]$$

Thus,

$$E_0 \frac{\partial}{\partial z} [\cos(kz - \omega t)] \hat{\mathbf{a}}_y = -\mu_0 H_0 \omega \sin(kz - \omega t) \hat{\mathbf{a}}_y$$

Therefore, we have

$$H_0 = \frac{kE_0}{\mu_0 \omega}$$

5. Answer the following briefly.

- (a) What is the frequency of your cell-phone? Calculate the corresponding wavelength in free-space.
- (b) What is the frequency of the power lines in Hong Kong? Calculate the corresponding wavelength in free-space.

Solution:

- (a) According to Office of the Communications Authority (OFCA) of Hong Kong (https://www.ofca.gov.hk/en/consumer_focus/guide/general/selection_of_5g_handsets/index.html), there are a number of bands used for mobile communications. For instance, for 5G, there is a band at 4.9 GHz, which corresponds a wavelength of:

$$\lambda = \frac{c}{4.0 \text{ GHz}} = 8.30 \text{ cm}$$

- (b) For Hong Kong, the A.C. power lines are operated at 50 Hz. Thus, the wavelength is

$$\lambda = \frac{c}{50 \text{ Hz}} = 6000 \text{ km}$$