

EE6617: Tutorial 1

1.1 For square and non-singular matrices $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times n}$, show that

- (a) $(\mathbf{A}\mathbf{b})^T = \mathbf{b}^T \mathbf{A}^T$ where $\mathbf{b} \in \mathbb{R}^n$.
- (b) $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T \mathbf{A}^T$.
- (c) $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$ (hint: use $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T \mathbf{A}^T$ and $\mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$).
- (d) $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1} \mathbf{A}^{-1}$ (hint: use $\mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$).
- (e) $\det(c\mathbf{A}) = c^n \det(\mathbf{A})$ (hint: use the definition of determinant).
- (f) $\det(\mathbf{I}) = 1$.
- (g) $\det(\mathbf{A}) = \prod_{i=1}^n a_{ii}$ for a diagonal matrix $\mathbf{A} = \text{diag}(a_{11}, a_{22}, \dots, a_{nn})$.
- (h) $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$ (hint: use $\det(\mathbf{A}\mathbf{B}) = \det(\mathbf{A}) \det(\mathbf{B})$).
- (i) $\text{tr}(\mathbf{A}\mathbf{B}) = \text{tr}(\mathbf{B}\mathbf{A})$.

1.2 Find the eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ associated with λ_1, λ_2 of $\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.
Is $\mathbf{A}$ is positive definite? What is $\text{tr}(\mathbf{A})$ and $\det(\mathbf{A})$? Show that

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T = [\mathbf{v}_1 \ \mathbf{v}_2] \text{diag}(\lambda_1, \lambda_2) [\mathbf{v}_1 \ \mathbf{v}_2]^T = [\mathbf{v}_2 \ \mathbf{v}_1] \text{diag}(\lambda_2, \lambda_1) [\mathbf{v}_2 \ \mathbf{v}_1]^T.$$

Is the eigenvector matrix $\mathbf{V}$ unique?

1.3 Find the eigenvalues and eigenvectors of the identity matrix $\mathbf{I} \in \mathbb{R}^{n \times n}$. Is the eigenvector matrix $\mathbf{V}$ is unique?

1.4 Let the basis of a vector space $\mathcal{S}$ be $\{\mathbf{x}_i\}_{i=1}^p$. Any vector $\mathbf{x} \in \mathcal{S}$ can be represented as a linear combination of the basis vectors. Show that the representation $\mathbf{x}$ is unique.

1.5 Suppose a set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

- (a) Find the basis of the span $\mathcal{S}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. Are they orthogonal? Are they independent?
- (b) Find $\dim(\mathcal{S}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3))$.

Now, suppose a matrix $\mathbf{X} = [\mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_3]$.

- (c) Find $\text{rank}(\mathbf{X})$ and $\dim(\mathcal{R}(\mathbf{X}))$.
- (d) Find $\text{rank}(\mathbf{X}^T)$ and $\dim(\mathcal{R}(\mathbf{X}^T))$.
- (e) Find any vectors in $\mathcal{N}(\mathbf{X})$.
- (f) Find $\dim(\mathcal{N}(\mathbf{X}))$ and $\dim(\mathcal{N}(\mathbf{X}^T))$.

Sol. 1.1 (a) Let the i th row of $\mathbf{A}$ be $\mathbf{a}_i$. Then, $(\mathbf{A}\mathbf{b})^T = \begin{bmatrix} \mathbf{a}_1\mathbf{b} \\ \vdots \\ \mathbf{a}_n\mathbf{b} \end{bmatrix}^T = [(\mathbf{a}_1\mathbf{b})^T (\mathbf{a}_2\mathbf{b})^T \cdots (\mathbf{a}_n\mathbf{b})^T] =$

$$[\mathbf{b}^T \mathbf{a}_1^T \cdots \mathbf{b}^T \mathbf{a}_n^T] = \mathbf{b}^T \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}^T = \mathbf{b}^T \mathbf{A}^T.$$

$$(b) \quad (\mathbf{A}\mathbf{B})^T = (\mathbf{A}[\mathbf{b}_1 \cdots \mathbf{b}_n])^T = ([\mathbf{A}\mathbf{b}_1 \cdots \mathbf{A}\mathbf{b}_n])^T = \begin{bmatrix} \mathbf{b}_1^T \mathbf{A}^T \\ \vdots \\ \mathbf{b}_n^T \mathbf{A}^T \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1^T \\ \vdots \\ \mathbf{b}_n^T \end{bmatrix} \mathbf{A}^T = \mathbf{B}^T \mathbf{A}^T$$

(c) $\mathbf{I} = \mathbf{A}\mathbf{A}^{-1} = (\mathbf{A}\mathbf{A}^{-1})^T = (\mathbf{A}^{-1})^T \mathbf{A}^T = \mathbf{I}$, implying $(\mathbf{A}^{-1})^T$ must be $(\mathbf{A}^T)^{-1}$, otherwise $(\mathbf{A}^{-1})^T \mathbf{A}^T \neq \mathbf{I}$.

(d) $\mathbf{I} = \mathbf{A}\mathbf{B}(\mathbf{A}\mathbf{B})^{-1} \Rightarrow \mathbf{A}^{-1} = \mathbf{B}(\mathbf{A}\mathbf{B})^{-1} \Rightarrow \mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{A}\mathbf{B})^{-1}$.

(e) By definition, $\det(c\mathbf{A}) = \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n ca_{i,\sigma_i} = c^n \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma_i} = c^n \det(\mathbf{A})$.

(f) For any diagonal matrix $\mathbf{A}$, we have $\det(\mathbf{A}) = \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma_i} = \text{sgn}(\{1, 2, \dots, n\}) \prod_{i=1}^n a_{i,i}$.

This is because $\prod_{i=1}^n a_{i,\sigma_i} = 0$ for $\sigma \neq \{1, 2, \dots, n\}$, i.e., the product $\prod_{i=1}^n a_{i,\sigma_i}$ always contains at least one $a_{i,\sigma_i} = 0$ when $\sigma \neq \{1, 2, \dots, n\}$ for a diagonal matrix. Substituting $\mathbf{A} = \mathbf{I}$ yields $\det(\mathbf{I}) = \text{sgn}(\{1, 2, \dots, n\}) \prod_{i=1}^n 1 = 1$.

(g) The same reasoning as in (e).

(h) Given $\det(\mathbf{A}\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})$, substitute $\mathbf{B} = \mathbf{A}^{-1}$ and use $\det(\mathbf{I}) = 1$.

(i) $\text{tr}(\mathbf{A}\mathbf{B}) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ji} = \sum_{i=1}^n \sum_{j=1}^n b_{ji}a_{ij} = \sum_{j=1}^n \sum_{i=1}^n b_{ji}a_{ij} = \text{tr}(\mathbf{B}\mathbf{A})$.

Sol. 1.2 First equate $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ and this is equivalent to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \Rightarrow \det(\mathbf{A} - \lambda\mathbf{I}) = 0$, i.e., the matrix $\mathbf{A} - \lambda\mathbf{I}$ should be singular. In general, the $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ yields an n th order polynomial equation (so-called the characteristic polynomial of $\mathbf{A}$), i.e., in our case,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \left(\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} \right) = (2+\lambda)^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda+1)(\lambda+3) = 0$$

Then, the roots of the above characteristic polynomial are the eigenvalues of $\mathbf{A}$, i.e., $\lambda_1 = -1$, $\lambda_2 = -3$.

For $\lambda_1 = -1$, the eigenvector $\mathbf{v}_1$ is found from $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = \mathbf{0}$, i.e.,

$$(\mathbf{A} + \mathbf{I})\mathbf{v}_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (1)$$

$$(2)$$

Similarly, for $\lambda_2 = -3$,

$$(\mathbf{A} + 3\mathbf{I}) \mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (3)$$

(4)

- We say if $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ for $\mathbf{x} \neq \mathbf{0}$, the matrix $\mathbf{A}$ is positive definite. For any $\mathbf{x} = [x_1 \ x_2]^T \in \mathbb{R}^{2 \times 1}$ and $\mathbf{A} = [\mathbf{v}_1 \ \mathbf{v}_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} [\mathbf{v}_1 \ \mathbf{v}_2]^T$, formulating $\mathbf{x}^T \mathbf{A} \mathbf{x} = \mathbf{x}^T [\mathbf{v}_1 \ \mathbf{v}_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} [\mathbf{v}_1 \ \mathbf{v}_2]^T \mathbf{x}$ returns $\frac{1}{2}(-(x_1 + x_2)^2 - 3(x_1 - x_2)^2) \leq 0$. Thus, it is not positive definite.
- $\text{tr}(\mathbf{A}) = \lambda_1 + \lambda_2 = -4$ and $\det(\mathbf{A}) = \lambda_1 \lambda_2 = 3$.
- $\mathbf{A} = [\mathbf{v}_1 \ \mathbf{v}_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} [\mathbf{v}_1 \ \mathbf{v}_2]^T = \sum_{i=1}^2 \lambda_i \mathbf{v}_i \mathbf{v}_i^T = [\mathbf{v}_2 \ \mathbf{v}_1] \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix} [\mathbf{v}_2 \ \mathbf{v}_1]^T$. Furthermore, it is easy to check if $\mathbf{V}$ is the eigenvector matrix of $\mathbf{A}$, then so is $-\mathbf{V}$. Thus, $\mathbf{V}$ is not unique.

Sol. 1.3 Followed by the same procedure, $(\mathbf{I} - \lambda \mathbf{I}) \mathbf{v} = \mathbf{0} \Rightarrow \det(\mathbf{I} - \lambda \mathbf{I}) = \det((1 - \lambda)\mathbf{I}) = (1 - \lambda)^n \det(\mathbf{I}) = 0 \Rightarrow (1 - \lambda)^n = 0$. The roots of the polynomial $(1 - \lambda)^n = 0$ are $\lambda_1 = \lambda_2 = \dots = \lambda_n = 1$. Since for any $\lambda_i = 1$, $(\mathbf{I} - \lambda_i \mathbf{I}) \mathbf{v}_i = \mathbf{0} \mathbf{v}_i = \mathbf{0}$, the eigenvector $\mathbf{v}_i$ associated with λ_i is not unique and can be any unit norm vector such that $\mathbf{v}_i^T \mathbf{v}_j = 0$ for $i \neq j$. Thus, the eigenvalue decomposition of $\mathbf{I}$ is

$$\mathbf{I} = \mathbf{V} \mathbf{I} \mathbf{V}^T,$$

with for any $\mathbf{V}$ such that $\mathbf{V} \mathbf{V}^T = \mathbf{V}^T \mathbf{V} = \mathbf{I}$.

Sol. 1.4 Since $\{\mathbf{x}_i\}_{i=1}^p$ forms the basis of the subspace $\mathcal{S}$, they are independent. Assume that there are two representations for a vector $\mathbf{x} \in \mathcal{S}$,

$$\mathbf{x} = \sum_{i=1}^p a_i \mathbf{x}_i = \sum_{i=1}^p b_i \mathbf{x}_i,$$

then

$$\sum_{i=1}^p a_i \mathbf{x}_i - \sum_{i=1}^p b_i \mathbf{x}_i = \sum_{i=1}^p (a_i - b_i) \mathbf{x}_i = \mathbf{0}.$$

Since the $\{\mathbf{x}_i\}_{i=1}^p$ are independent, it follows that $a_i = b_i$, $i = 1, \dots, p$, meaning the representation of $\mathbf{x}$ is unique.

Sol. 1.5 (a) Basis = $\{\mathbf{x}_1, \mathbf{x}_2\}$. It is easy to check $\mathbf{x}_3 = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$. The vectors $\mathbf{x}_1, \mathbf{x}_2$ are not orthogonal, but are independent.

(b) $\dim(\mathcal{S}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)) = 2$.

(c) $\dim(\mathcal{S}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)) = \text{rank}(\mathbf{X}) = 2$. $\dim(\mathcal{S}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)) = \dim(\mathcal{R}(\mathbf{X})) = 2$.

(d) Form $\mathbf{X}^T$,

$$\mathbf{X}^T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

It is easy to see the 2nd and 3rd columns are dependent. Hence $\text{rank}(\mathbf{X}^T) = 2$ and thereby $\dim(\mathcal{R}(\mathbf{X}^T)) = 2$.

(e) Choose $\mathbf{a} = [1 \ 1 \ -2]^T$. Then, $\mathbf{X}\mathbf{a} = \mathbf{0}$ and hence $\mathbf{a} \in \mathcal{N}(\mathbf{X})$.

(f) $\dim(\mathcal{N}(\mathbf{X})) = 3 - \dim(\mathcal{R}(\mathbf{X})) = 1$ and $\dim(\mathcal{N}(\mathbf{X}^T)) = 3 - \dim(\mathcal{R}(\mathbf{X}^T)) = 1$