

EF2452: Mathematics for Economics and Finance

Lecture 4

Part II of II

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Main Topics

- Elasticity of demand
- Profit maximization: $MR = MC$ rule
- A monopolist's profit maximization

Elasticity of demand

- Consider the demand function $q(p)$. Then the **elasticity of demand**, ε at p_1 is defined as follows

$$\varepsilon_{p_1} = \frac{\% \text{change in quantity demanded}}{\% \text{change in price}}$$

$$\varepsilon_{p_1} = - \frac{\left(\frac{q_2 - q_1}{q_1} \right)}{\left(\frac{p_2 - p_1}{p_1} \right)} = - \frac{\left(\frac{\Delta q}{q_1} \right)}{\left(\frac{\Delta p}{p_1} \right)} = - \frac{\Delta q}{\Delta p} \frac{p_1}{q_1}$$

- We put a negative sign in front because (usually) price and quantity demanded are negatively related so that an increase in price results in a decrease in quantity demanded, and we want to have the measure of elasticity to have a positive value.

Elasticity of demand

- Suppose that the demand function $q(p)$ is differentiable. Then the **elasticity of demand**, $\varepsilon(p)$ is defined as

$$\varepsilon(p) = -\frac{q'p}{q} = -\frac{dq}{dp} \frac{p}{q}$$

Elasticity of demand: An Example

- Suppose the demand set is

$$D = \{(q, p) | q + 5p = 30\}$$

- Then the demand function is: $q(p) = 30 - 5p$.
- The elasticity of demand is given by

$$\varepsilon(p) = -\frac{q'p}{q} = -\frac{(-5)p}{30 - 5p} = \frac{p}{6 - p}$$

- $\varepsilon(p)$ is not defined when $p = 6$ (as the denominator is zero when $p = 6$).

Elasticity of demand: An Example

- Restricting ourselves to economically meaningful values for price, that is, the values of p which both p and q are non-negative, we need the restriction: $0 \leq p \leq 6$.
- The demand is elastic when

$$\varepsilon(p) > 1 \implies \frac{p}{6-p} > 1 \implies 3 < p < 6$$

- The demand is inelastic when

$$\varepsilon(p) < 1 \implies \frac{p}{6-p} < 1 \implies 0 \leq p < 3$$

- The demand is unitary elastic at $p = 3$.

Elasticity of demand and Revenue

- Noting that quantity is a function of price, we can write the revenue as a function of price, i.e., $R(p) = pq = pq(p)$. Differentiating $R(p)$ w.r.t. p , we get

$$\begin{aligned} R'(p) &= q(p) + pq'(p) \\ &= q(p) \left[1 + \frac{pq'(p)}{q(p)} \right] \\ &= q(p)(1 - \varepsilon(p)) \end{aligned}$$

- If demand is inelastic ($\varepsilon(p) < 1$), $R'(q) > 0$: a small increase in price results in an increase in revenue.
- If demand is elastic ($\varepsilon(p) > 1$), $R'(q) < 0$: a small increase in price results in a decrease in revenue.

Profit maximization: $MR = MC$ rule

- The profit of a firm is given by

$$\Pi(q) = R(q) - C(q)$$

- Suppose $q \in [0, L]$ and that there is an interior solution: profit is maximized at $0 < q_m < L$ with

$$\Pi'(q_m) = 0 \quad \text{that is,} \quad R'(q_m) = C'(q_m)$$

- In other words, at the optimum production level, the marginal revenue (the revenue resulting from producing one more unit) to equal the marginal cost (the cost of producing one more unit).

Profit maximization: $MR = MC$ rule

- If the cost of producing one extra unit is less than the revenue from that additional unit, the extra unit should be produced: the firm should produce more.

$$MR > MC \implies q \uparrow$$

- If the extra cost exceeds the extra revenue from an additional unit of production, the firm should produce less.

$$MR < MC \implies q \downarrow$$

A monopolist's profit maximization

- Consider the case of a monopoly, where there is a single firm supplying the entire quantity of the good in a market.
- $P(q)$ is the inverse demand function: the price the consumers will pay when q is available.
- The profit function of the monopolist is

$$\Pi(q) = qP(q) - C(q)$$

- The profit maximization condition $\Pi'(q) = 0$ becomes

$$qP'(q) + P(q) - C'(q) = 0$$

$$qP'(q) + P(q) = C'(q)$$

- We will generally have an interval for q : $[0, L]$. We must also check for (a) second-order conditions (sufficient conditions) for maxima and (b) corner solutions for the maximization exercise.

Monopolist's profit maximization: An Example

- Suppose the only firm manufacturing a certain kind of machine tool can produce up to 100 per week. The demand set for these items is

$$D = \{(q, p) | q + 5p = 850\}$$

- The cost of producing q items per week is

$$C(q) = 300 - 10q + q^2$$

- How many items should be produced each week in order to maximize profit?

Monopolist's profit maximization: An Example

- The monopolist faces the following inverse demand:

$$P(q) = \frac{850 - q}{5} = 170 - 0.2q$$

- The weekly profit from producing q machine tools is

$$\begin{aligned}\Pi(q) &= qP(q) - C(q) = q(170 - 0.2q) - (300 - 10q + q^2) \\ &= -1.2q^2 + 180q - 300\end{aligned}$$

- The first order condition ($\Pi'(q) = 0$) is

$$-2.4q + 180 = 0$$

- Thus, there is one critical point $q_m = 75$. Note that $\Pi(75) = 6450$.

Monopolist's profit maximization: An Example

- In order to be sure that this is indeed the optimum value, we have to check two things.
- First, would it not be better to produce either zero (or nothing) or the upper limit of 100 items per week?
- $\Pi(0) = -300$ and $\Pi(100) = 5700$; therefore, neither of these possibilities is relevant.
- Secondly, is the second-order condition satisfied for a maximum?
- We have $\Pi''(75) = -2.4 < 0$; so, the condition is satisfied.

References

Following chapter of the textbook:

9 The Derivative in Economics I