

EE3109 Applied Electromagnetics
2024/25 Semester B
Week 2 Tutorial Solutions

1. Given $\mathbf{E}(t) = A_0 \sin(\omega t - \beta z + \phi) \hat{\mathbf{x}}$, find $\mathbf{H}(t)$ using phasors.

Solution:

Using Phasor notation,

$$\mathbf{E} = A_0 e^{-j\beta z} e^{j(\phi - \frac{\pi}{2})} \hat{\mathbf{x}} = E_0 \hat{\mathbf{x}}$$

Using Faraday's Law:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

Let $E_0 = A_0 e^{-j\beta z} e^{j\phi}$, s.t. $\mathbf{E} = E_0 \hat{\mathbf{x}}$, Then:

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0 & 0 & 0 \end{vmatrix} = \frac{\partial E_0}{\partial z} \hat{\mathbf{y}} = \frac{\partial A_0 e^{-j\beta z} e^{j(\phi - \frac{\pi}{2})}}{\partial z} \hat{\mathbf{y}} = -j\beta A_0 e^{-j\beta z} e^{j(\phi - \frac{\pi}{2})} \hat{\mathbf{y}}$$

Therefore;

$$\begin{aligned} -j\omega\mu\mathbf{H} &= -j\beta A_0 e^{-j\beta z} e^{j(\phi - \frac{\pi}{2})} \hat{\mathbf{y}} \\ \Rightarrow \mathbf{H} &= -\frac{j\beta A_0 e^{-j\beta z} e^{j(\phi - \frac{\pi}{2})}}{-j\omega\mu} \hat{\mathbf{y}} = \frac{\beta}{\omega\mu} A_0 e^{-j\beta z} e^{j(\phi - \frac{\pi}{2})} \hat{\mathbf{y}} = \frac{\beta}{\omega\mu} E_0 \hat{\mathbf{y}} \end{aligned}$$

Now convert back into time domain:

$$\mathbf{H}(t) = \text{Re}\{\mathbf{H}e^{j\omega t}\} = \text{Re}\left\{\frac{\beta}{\omega\mu} A_0 e^{j(\omega t - \beta z + \phi - \frac{\pi}{2})}\right\} \hat{\mathbf{y}} = \frac{\beta}{\omega\mu} A_0 \sin(\omega t - \beta z + \phi) \hat{\mathbf{y}} \text{ [A/m]}$$

NB: (i) Differentiation in phasor corresponds to multiplication by $j\omega$;

(ii) Integration in phasor corresponds to multiplication by $\frac{1}{j\omega}$.

2. Given $\mathbf{H}(t) = A_0 \cos(\omega t - kx - ky + \phi) \hat{\mathbf{z}}$, find $\mathbf{E}(t)$ using phasors.

Solution:

Using Phasor notation,

$$\mathbf{H} = A_0 e^{-j(kx+ky)} e^{j\phi} \hat{\mathbf{z}} = H_0 \hat{\mathbf{z}}$$

Using Ampere's Law:

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$$

Let $H_0 = A_0 e^{-j(kx+ky)} e^{j\phi}$, s.t. $\mathbf{H} = H_0 \hat{\mathbf{z}}$, Then:

$$\begin{aligned} \nabla \times \mathbf{H} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_0 \end{vmatrix} = \frac{\partial H_0}{\partial y} \hat{\mathbf{x}} - \frac{\partial H_0}{\partial x} \hat{\mathbf{y}} = \frac{\partial (A_0 e^{-j(kx+ky)} e^{j\phi})}{\partial y} \hat{\mathbf{x}} - \frac{\partial (A_0 e^{-j(kx+ky)} e^{j\phi})}{\partial x} \hat{\mathbf{y}} \\ &= jkA_0 e^{-j(kx+ky)} e^{j\phi} (\hat{\mathbf{y}} - \hat{\mathbf{x}}) \end{aligned}$$

Therefore;

$$\begin{aligned} j\omega\epsilon\mathbf{E} &= jkA_0 e^{-j(kx+ky)} e^{j\phi} (\hat{\mathbf{y}} - \hat{\mathbf{x}}) \\ \Rightarrow \mathbf{E} &= \frac{jkA_0 e^{-j(kx+ky)} e^{j\phi}}{j\omega\epsilon} (\hat{\mathbf{y}} - \hat{\mathbf{x}}) = \frac{k}{\omega\epsilon} A_0 e^{-j(kx+ky)} e^{j\phi} (\hat{\mathbf{y}} - \hat{\mathbf{x}}) = \frac{k}{\omega\epsilon} H_0 (\hat{\mathbf{y}} - \hat{\mathbf{x}}) \end{aligned}$$

Now convert back into time domain:

$$\mathbf{E}(t) = \text{Re}\{\mathbf{E}e^{j\omega t}\} = \text{Re}\left\{\frac{k}{\omega\epsilon}A_0e^{j(\omega t - kx - ky + \phi)}\right\}(\hat{\mathbf{y}} - \hat{\mathbf{x}}) = \frac{k}{\omega\epsilon}A_0\cos(\omega t - kx - ky + \phi)(\hat{\mathbf{y}} - \hat{\mathbf{x}})$$

3. When observed at large distances from their sources, the electric and magnetic fields by many current and charge distributions can be approximated as spherically symmetric waves whose amplitudes vary inversely with distance. Consider the following spherical wave in source-free, free space:

$$\mathbf{E} = \frac{E_0}{r}\sin(\omega t - \beta r)\hat{\mathbf{a}}_\theta$$

- Use Maxwell's curl- $\mathbf{E}$ equation to find the corresponding magnetic field intensity $\mathbf{H}$.
- Substitute the $\mathbf{H}$ found in (a) into Maxwell's curl- $\mathbf{H}$ equation to find the values of β that allows $\mathbf{E}$ to satisfy Maxwell's curl equations for large values of r (far-field approximation).
- What other component of $\mathbf{E}$ must also be present at small values of r in order to satisfy Maxwell's equations?

Solution:

- (a) We were given ;

$$\mathbf{E} = \frac{E_0}{r}\sin(\omega t - \beta r)\hat{\mathbf{a}}_\theta$$

Recall that;

$$Ae^{j\theta} = A(\cos\theta + j\sin\theta)$$

Thus;

$$\mathbf{E} = \frac{E_0}{r}\sin(\omega t - \beta r)\hat{\mathbf{a}}_\theta = \frac{E_0}{r}e^{-j(\beta r + \frac{\pi}{2})}\hat{\mathbf{a}}_\theta = E_\theta\hat{\mathbf{a}}_\theta$$

Note that $E_r = E_\phi = 0$

The Curl of $\mathbf{E}$ in spherical coordinate is computed as below;

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ \frac{1}{r^2\sin\theta} \frac{\partial}{\partial r} & \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta} \frac{\partial}{\partial \phi} \\ E_r & rE_\theta & r\sin\theta E_\phi \end{vmatrix}$$

$$\nabla \times \mathbf{E} = \frac{1}{r^2\sin\theta} \left(\frac{\partial}{\partial \phi} rE_\theta \right) \hat{\mathbf{r}} + \frac{1}{r} \left(\frac{\partial}{\partial r} rE_\theta \right) \hat{\boldsymbol{\phi}} = 0\hat{\mathbf{r}} - j \frac{E_0\beta}{r} e^{-j(\beta r + \frac{\pi}{2})} \hat{\boldsymbol{\phi}}$$

$$\mathbf{H} = \frac{-1}{j\omega\mu_0} \nabla \times \mathbf{E} = \left(\frac{E_0\beta}{\omega\mu_0} \right) \left(\frac{1}{r} \right) e^{-j(\beta r + \frac{\pi}{2})} = H_\phi \hat{\boldsymbol{\phi}}$$

Converting from phasor notation gives;

$$\mathbf{H} = \left(\frac{E_0\beta}{\omega\mu_0} \right) \left(\frac{1}{r} \right) \sin(\omega t - \beta r) \hat{\boldsymbol{\phi}} \text{ [A/m]}$$

- (b) The procedure is similar to what we did above

$$\begin{aligned} \nabla \times \mathbf{H} &= \frac{1}{r\sin\theta} \left(\frac{\partial}{\partial \theta} H_\phi \sin\theta \right) \hat{\mathbf{r}} - \frac{1}{r} \left(\frac{\partial}{\partial r} rH_\phi \right) \hat{\boldsymbol{\theta}} \\ &= \left(\frac{\beta E_0 \cot\theta}{\omega\mu_0} \right) \frac{1}{r^2} e^{-j(\beta r + \frac{\pi}{2})} \hat{\mathbf{r}} + j \left(\frac{\beta^2 E_0}{\omega\mu_0} \right) \frac{1}{r} e^{-j(\beta r + \frac{\pi}{2})} \hat{\boldsymbol{\theta}} \end{aligned}$$

Applying the curl-H equation;

$$\mathbf{E} = \frac{1}{j\omega\epsilon_0} \nabla \times \mathbf{H} = -j \left(\frac{\beta E_0 \cot\theta}{\omega^2\mu_0\epsilon_0} \right) \frac{1}{r^2} e^{-j(\beta r + \frac{\pi}{2})} \hat{\mathbf{r}} + \left(\frac{\beta^2 E_0}{\omega^2\mu_0\epsilon_0} \right) \frac{1}{r} e^{-j(\beta r + \frac{\pi}{2})} \hat{\boldsymbol{\theta}} = E_{r.new} \hat{\mathbf{r}} + E_{\theta.new} \hat{\boldsymbol{\theta}}$$

For large values of r , $\mathbf{E}_{new} \approx E_{\theta.new} \hat{\boldsymbol{\theta}} \Rightarrow \beta^2 = \omega^2\mu_0\epsilon_0$

Note: the fact that $\cot\theta \rightarrow \infty$ when $\theta \rightarrow m\pi$ means that the $\hat{\mathbf{r}}$ component is non-negligible (or even dominant) when θ is close to $m\pi$. A contradiction thus arises since in the original question $E_r = 0$. This contradiction arises most likely because the electric field in the question does not really exist in a source-less free-space environment. Nonetheless, in regions where θ is not close to $m\pi$, and particularly for θ

close to 90° , the approximation made above is true and the relationship obtained above is valid. This relation will manifest itself in the lecture on plane waves.

(c) For small values of r , the $\hat{r}$ component of $\mathbf{E}$ must also be present to satisfy Maxwell's equations.

4. A homogenous dielectric with dielectric constant $\epsilon_r = 8$ and conductivity σ is placed between the plates of a parallel-plate capacitor. Determine the value σ at which the conduction and displacement currents in the dielectric are equal at: (a) $f = 1\text{kHz}$, (b) $f = 100\text{MHz}$

Solution:

Given $\epsilon_r = 8$

The currents in phasor form are written as;

$$\mathbf{J}_{cond} = \sigma \mathbf{E}$$

$$\mathbf{J}_{disp} = j\omega\epsilon\mathbf{E}$$

For both currents to be equal (in amplitude), it is required that;

$$|\mathbf{J}_{cond}| = |\mathbf{J}_{disp}|$$

$$|\sigma \mathbf{E}| = |j\omega\epsilon\mathbf{E}|$$

$$\sigma = \omega\epsilon = 2\pi f\epsilon_r\epsilon_0 \quad (1)$$

Using equation (1);

(a) At $f = 1\text{kHz}$, $\sigma = 4.45 \times 10^{-7} \text{ S/m}$

(b) At $f = 100\text{MHz}$, $\sigma = 0.0445 \text{ S/m}$

5. The parallel-plate capacitor shown in Fig. 1 is filled with a lossy dielectric material of relative permittivity ϵ_r and conductivity σ . The separation between the plates is d and each plate is of area A . The capacitor is connected to a time-varying voltage source $V(t)$
- Obtain an expression for I_c , the conduction current flowing between the plates inside the capacitor, in terms of the given quantities.
 - Obtain an expression for I_d , the displacement current flowing inside the capacitor.
 - Based on your expressions for parts (a) and (b), give an equivalent circuit representation for the capacitor.
 - Evaluate the values of the circuit elements for $A = 4\text{cm}^2$, $d = 0.5 \text{ [cm]}$, $\epsilon_r = 4$, $\sigma = 2.5 \text{ [S/m]}$ and $V(t) = 10 \cos(3\pi \times 10^3 t) \text{ [V]}$

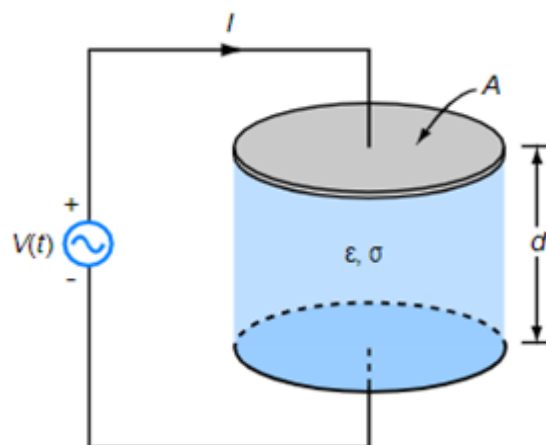


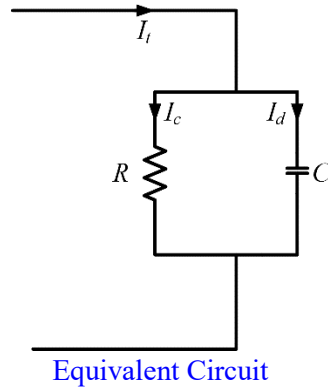
Fig. 1. Parallel-plate capacitor containing a lossy dielectric

Solution:

(a) Recall that $R = \frac{d}{\sigma A}$. Thus, $I_c = \frac{V}{R} = \frac{V\sigma A}{d}$

(b) Recall that $E = \frac{V}{d}$ and $D = \epsilon E$. Then, $I_d = J_d A = \frac{\partial D}{\partial t} A = \epsilon \frac{\partial E}{\partial t} A = \frac{\epsilon A}{d} \frac{\partial V}{\partial t}$.

- (c) The conduction current is directly proportional to V , as characteristic of a resistor, whereas the displacement current varies as $\frac{\partial V}{\partial t}$ which is characteristic of a capacitor. Hence, this is a parallel RC circuit with $R = \frac{d}{\sigma A}$ and $C = \frac{\epsilon A}{d}$.



(d) $R = \frac{d}{\sigma A} = \frac{0.5 \times 10^{-2}}{2.5 \times 4 \times 10^{-4}} = 5 \text{ } [\Omega]$; $C = \frac{\epsilon A}{d} = 2.84 \text{ pF}$.

6. If the current density in a conducting medium is given by $\mathbf{J}(x, y, z; t) = (\hat{x}z - \hat{y}4y^2 + \hat{z}2x) \cos \omega t$, determine the corresponding charge distribution $\rho_v(x, y, z; t)$.

Solution:

Recall that;

$$\begin{aligned} \nabla \cdot \mathbf{J} &= -\frac{\partial \rho_v}{\partial t} \\ \nabla \cdot \mathbf{J} &= \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot (\hat{x}z - \hat{y}4y^2 + \hat{z}2x) \cos \omega t \\ &= -4 \frac{\partial}{\partial t} (y^2 \cos \omega t) = -8y \cos \omega t \end{aligned}$$

Substituting this result back into equation (1) and integrating both sides w.r.t. t gives;

$$\rho_v = - \int (\nabla \cdot \mathbf{J}) dt = - \int -8y \cos \omega t \, dt = \frac{8y}{\omega} \sin \omega t + C_0$$

Where C_0 is a constant of integration.

7. The magnetic field in a given dielectric medium is given by $\mathbf{H} = 6 \cos 2z \sin(2 \times 10^7 t - 0.1x) \hat{\mathbf{y}}$ [A/m], where x and z are in meters. Determine:
- The electric field $\mathbf{E}$,
 - The displacement current density $\mathbf{J}_d$, and
 - The charge density ρ_v .

Solution:

(a) Given;

$$\begin{aligned}\tilde{\mathbf{H}} &= \hat{\mathbf{y}} 6 \cos 2z e^{-j0.1x} e^{-j\pi/2} = -\hat{\mathbf{y}} j 6 \cos 2z e^{-j0.1x} \\ \tilde{\mathbf{E}} &= \frac{1}{j\omega\epsilon} \nabla \times \tilde{\mathbf{H}} \\ &= \frac{1}{j\omega\epsilon} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & -j6 \cos 2z e^{-j0.1x} & 0 \end{vmatrix} \\ &= \frac{1}{j\omega\epsilon} \left\{ \hat{\mathbf{x}} \left[-\frac{\partial}{\partial z} (-j6 \cos 2z e^{-j0.1x}) \right] + \hat{\mathbf{z}} \left[\frac{\partial}{\partial x} (-j6 \cos 2z e^{-j0.1x}) \right] \right\} \\ &= \hat{\mathbf{x}} \left(-\frac{12}{\omega\epsilon} \sin 2z e^{-j0.1x} \right) + \hat{\mathbf{z}} \left(\frac{j0.6}{\omega\epsilon} \cos 2z e^{-j0.1x} \right)\end{aligned}$$

From the given expression for $\mathbf{H}$, $\omega = 2 \times 10^7$ (rad/s) and $k = 0.1$ (rad/m)
Since the material is dielectric, $\mu = \mu_0 = 4\pi \times 10^{-7}$ (H/m)

So,

$$k^2 = \omega^2 \mu \epsilon \Rightarrow \epsilon = \frac{k^2}{\omega^2 \mu_0} = \frac{0.1^2}{(2 \times 10^7)^2 (4\pi \times 10^{-7})} = 1.989 \times 10^{-11} \text{ (F/m)}$$

Using the values for ω and ϵ , we have

$$\begin{aligned}\tilde{\mathbf{E}} &= (-\hat{\mathbf{x}} 30 \sin 2z + \hat{\mathbf{z}} j 1.5 \cos 2z) \times 10^3 e^{-j0.1x} \text{ (V/m)} \\ \mathbf{E} &= [-\hat{\mathbf{x}} 30 \sin 2z \cos(2 \times 10^7 t - 0.1x) - \hat{\mathbf{z}} 1.5 \cos 2z \sin(2 \times 10^7 t - 0.1x)] \text{ (kV/m)}\end{aligned}$$

(b)

$$\begin{aligned}\tilde{\mathbf{D}} &= \epsilon \tilde{\mathbf{E}} = \epsilon_r \epsilon_0 \tilde{\mathbf{E}} = (-\hat{\mathbf{x}} 0.6 \sin 2z + \hat{\mathbf{z}} j 0.03 \cos 2z) \times 10^{-6} e^{-j0.1x} \text{ (C/m}^2\text{)} \\ \mathbf{J}_d &= \frac{\partial \tilde{\mathbf{D}}}{\partial t}\end{aligned}$$

Or

$$\begin{aligned}\tilde{\mathbf{J}}_d &= j\omega \tilde{\mathbf{D}} = (-\hat{\mathbf{x}} j 12 \sin 2z - \hat{\mathbf{z}} j 0.6 \cos 2z) e^{-j0.1x} \\ \mathbf{J}_d &= \Re \{ \tilde{\mathbf{J}}_d e^{j\omega t} \} \\ &= [-\hat{\mathbf{x}} 12 \sin 2z \sin(2 \times 10^7 t - 0.1x) - \hat{\mathbf{z}} 0.6 \cos 2z \cos(2 \times 10^7 t - 0.1x)] \text{ (A/m}^2\text{)}\end{aligned}$$

(c) We can find ρ_v from

$$\nabla \cdot \mathbf{D} = \rho_v$$

or from

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

Applying Maxwell's equation,

$$\rho_v = \nabla \cdot \mathbf{D} = \epsilon \nabla \cdot \mathbf{E} = \epsilon_r \epsilon_0 \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} \right)$$

Therefore;

$$\begin{aligned}\rho_v &= \epsilon_r \epsilon_0 \left\{ \frac{\partial}{\partial x} [-30 \sin 2z \cos(2 \times 10^7 t - 0.1x)] + \frac{\partial}{\partial z} [-1.5 \cos 2z \sin(2 \times 10^7 t - 0.1x)] \right\} \\ &= \epsilon_r \epsilon_0 [-3 \sin 2z \sin(2 \times 10^7 t - 0.1x) + 3 \sin 2z \sin(2 \times 10^7 t - 0.1x)] = 0.\end{aligned}$$