

1. Change the order of the integration in $\int_0^1 \left[\int_0^{2y} f(x, y) dx \right] dy + \int_1^3 \left[\int_0^{3-y} f(x, y) dx \right] dy$.

Solution:

Let R be region of the integration. Observe that R is bounded by $x = 0$, $x = 2y$, $x = 3 - y$.

$$\begin{cases} x = 2y \\ x = 3 - y \end{cases} \Rightarrow 2y = 3 - y \Leftrightarrow y = 1. \text{ And } \begin{cases} y = 1 \\ x = 3 - y \end{cases} \Rightarrow x = 2.$$

It follows that $x = 2y$, $x = 3 - y$ intercept at the point $(2, 1)$.

$$\int_0^1 \left[\int_0^{2y} f(x, y) dx \right] dy + \int_1^3 \left[\int_0^{3-y} f(x, y) dx \right] dy = \iint_R f(x, y) dx dy = \int_0^2 \left[\int_{\frac{x}{2}}^{3-x} f(x, y) dy \right] dx.$$

2. Evaluate the following double integrals:

(a) $\iint_S xy \, dx dy$, where S is the region bounded by the lines $x = 0$, $x = 1$, $y = x^2$ and $y = 4$.

(b) $\iint_S x^2 \, dx dy$, where S is the region bounded by $y = 2x$ and $x^2 + y = 8$.

Sol:

(a)

$$\iint_S xy \, dx dy = \int_0^1 \left[\int_{x^2}^4 xy dy \right] dx = \int_0^1 \frac{xy^2}{2} \Big|_{x^2}^4 dx = \int_0^1 \frac{16x - x^5}{2} dx = \left[4x^2 - \frac{x^6}{12} \right]_0^1 = 4 - \frac{1}{12} = \frac{47}{12}$$

(b)

$$\begin{cases} y = 2x \\ x^2 + y = 8 \end{cases} \Leftrightarrow \begin{cases} y = 2x \\ x^2 + 2x - 8 = 0 \end{cases} \Leftrightarrow \begin{cases} y = 2x \\ (x+4)(x-2) = 0 \end{cases} \Leftrightarrow \begin{cases} y = 2x \\ x = -4 \text{ or } x = 2 \end{cases} \Leftrightarrow \begin{cases} x = -4 \\ y = -8 \end{cases} \text{ or } \begin{cases} x = 2 \\ y = 4 \end{cases}$$

$$\begin{aligned} \iint_S x^2 \, dx dy &= \int_{-4}^2 \left[\int_{2x}^{8-x^2} x^2 dy \right] dx = \int_{-4}^2 x^2 y \Big|_{2x}^{8-x^2} dx = \int_{-4}^2 x^2 (8 - x^2 - 2x) dx = \left[\frac{8x^3}{3} - \frac{x^5}{5} - \frac{x^4}{2} \right]_{-4}^2 \\ &= \frac{64}{3} - \frac{32}{5} - 8 - \left(-\frac{512}{3} - \frac{1024}{5} - 128 \right) = 192 - 211.2 + 120 = 100.8 \end{aligned}$$

- *3. Evaluate $\iint_S xy \, dx dy$ where S is the region enclosed by the 4 parabolas $y^2 = x$, $y^2 = 2x$, $x^2 = y$,

$$x^2 = 2y \text{ using the change of variable } u = \frac{x^2}{y}, v = \frac{y^2}{x}.$$

Sol:

Under the transformations $u = \frac{x^2}{y}, v = \frac{y^2}{x}$, the boundaries $y^2 = x$ goes to $v = 1$,

$y^2 = 2x$ to $v = 2$, $x^2 = y$ to $u = 1$ and $x^2 = 2y$ to $u = 2$. And the region S in x - y plane which are enclosed by the 4 parabolas $y^2 = x$, $y^2 = 2x$, $x^2 = y$, $x^2 = 2y$ goes to the region $\bar{S}$ in u - v plane which are enclosed by the lines $u = 1, u = 2, v = 1, v = 2$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}} = \frac{1}{\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}} = \frac{1}{\det \begin{pmatrix} \frac{2x}{y} & -\frac{y^2}{x^2} \\ \frac{y}{x^2} & \frac{2y}{x} \end{pmatrix}} = \frac{1}{3}$$

Therefore,

$$\iint_S xy \, dxdy = \iint_{\bar{S}} uv |J| \, dudv = \int_1^2 \left[\int_1^2 \frac{1}{3} uv dv \right] du = \int_1^2 \frac{1}{2} u du = \frac{3}{4}$$

4. Evaluate $E_z = \frac{\sigma_0 z}{4\pi \epsilon_0} \iint_S \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \, dxdy$ where S is the disc $x^2 + y^2 \leq a^2$, which represents the z -component of the electric field at the point $(0, 0, z)$ due to a uniformly charged circular disc lying in $x^2 + y^2 \leq a^2, z = 0$.

Sol:

$$x^2 + y^2 = r^2, dxdy = r dr d\theta$$

$$\begin{aligned} E_z &= \frac{\sigma_0 z}{4\pi \epsilon_0} \iint_S \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \, dxdy = \frac{\sigma_0 z}{4\pi \epsilon_0} \int_0^a \left[\int_0^{2\pi} \frac{r d\theta}{(r^2 + z^2)^{3/2}} \right] dr \\ &= \frac{2\pi \sigma_0 z}{4\pi \epsilon_0} \int_0^a \frac{r dr}{(r^2 + z^2)^{3/2}} = \frac{\sigma_0 z}{2 \epsilon_0} \left. \frac{-1}{(r^2 + z^2)^{1/2}} \right|_0^a = -\frac{\sigma_0 z}{2 \epsilon_0} \left(\frac{1}{(a^2 + z^2)^{1/2}} - \frac{1}{|z|} \right) \end{aligned}$$

- *5. Let R be the region bounded by $x + y = 1, x = 0, y = 0$. Show that

$$\iint_R \cos\left(\frac{x-y}{x+y}\right) \, dxdy = \frac{\sin 1}{2}, \text{ using the substitution } x-y=u, x+y=v.$$

Sol:

$$x+y=1 \rightarrow v=1, x=0 \rightarrow u+v=0, y=0 \rightarrow u-v=0$$

$$\begin{aligned} x-y=u \\ x+y=v \end{aligned} \rightarrow \begin{aligned} x &= \frac{u+v}{2} \\ y &= \frac{v-u}{2} \end{aligned}, \text{ Jacobian} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = 1/2$$

$$\iint_R \cos\left(\frac{x-y}{x+y}\right) dx dy = \iint_{R'} \frac{1}{2} \cos\left(\frac{u}{v}\right) du dv = \int_0^1 \int_{-v}^v \frac{1}{2} \cos\left(\frac{u}{v}\right) du dv = \frac{\sin 1}{2}$$

*6. Evaluate $\iint_S e^{xy} dx dy$ where S is the region enclosed by $xy=1$, $xy=2$, $y=x$, $y=4x$ using the change of variable $xy=u$, $\frac{y}{x}=v$.

Sol:

Under the transformations $xy=u$, $\frac{y}{x}=v$,

$xy=1$ goes to $u=1$, $xy=2$ goes to $u=2$, $y=x$ goes to $v=1$, $y=4x$ goes to $v=4$.

The region S in x - y plane which is enclosed by $xy=1$, $xy=2$, $y=x$, $y=4x$ consists of two parts, say S_1 & S_2 , that is, $S = S_1 \cup S_2$. S_1 is in the first quadrant of xy -plane and S_2 in the third quadrant. Under the transformations $xy=u$, $\frac{y}{x}=v$, both S_1 & S_2 go to the region $\bar{S}$ in u - v plane which is enclosed by the lines $u=1, u=2, v=1, v=4$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}} = \frac{1}{\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}} = \frac{1}{\det \begin{pmatrix} y & -\frac{y}{x^2} \\ x & \frac{1}{x} \end{pmatrix}} = \frac{1}{\frac{y}{x} + \frac{y}{x}} = \frac{1}{2v}$$

$$\iint_S e^{xy} dS = 2 \iint_{\bar{S}} e^u \left| \frac{1}{2v} \right| du dv = 2 \int_1^2 \left[\int_1^4 e^u \frac{1}{2v} dv \right] du = \int_1^2 (e^u \ln v) \Big|_1^4 du = \int_1^2 (\ln 4) e^u du = (e^2 - e) \ln 4$$

*7. Use the change of variables $x+y=u$, $x-y=v$ to evaluate $\iint_{|x|+|y|\leq 1} e^{(x-y)} dx dy$.

Sol:

$|x|+|y|\leq 1$ is equivalent to the region bounded by $x+y=1, -x-y=1, -x+y=1, x-y=1$.

Under the transformations: $x+y=u$, $x-y=v$,

$x+y=1$ goes to $u=1$, $-x-y=1$ goes to $u=-1$, $-x+y=1$ goes to $v=-1$ and $x-y=1$ goes to $v=1$.

$$\text{The Jacobian } J = \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}} = \frac{1}{\det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} = -\frac{1}{2}.$$

So

$$\iint_{|x|+|y|\leq 1} e^{(x-y)} dx dy = \iint_{\substack{-1\leq x\leq 1 \\ -1\leq y\leq 1}} e^v |J| du dv = \int_{-1}^1 \left[\int_{-1}^1 \frac{e^v}{2} dv \right] du = \int_{-1}^1 \frac{e^v}{2} \Big|_{-1}^1 du = \frac{1}{2} \int_{-1}^1 (e - e^{-1}) du = e - e^{-1}.$$

8. An iterated integral like $\int_0^1 \left[\int_0^{\frac{1-x}{2}} \left(\int_0^{1-x-2y} f(x, y, z) dz \right) dy \right] dx$ is called an iterated integral with order $dzdydx$.

Change the order of the iterated integral $\int_0^1 \left[\int_0^{\frac{1-x}{2}} \left(\int_0^{1-x-2y} f(x, y, z) dz \right) dy \right] dx$ to an equivalent iterated integral with order $dx dz dy$.

Solution:

Suppose we evaluate the triple integral of the function $f(x, y, z)$ in a solid V .

Let the projection of V onto x - y plane is σ_{xy} .

σ_{xy} is the set $\left\{ (x, y) : 0 \leq x \leq 1, 0 \leq y \leq \frac{1-x}{2} \right\}$, which is the triangle enclosed by $x = 0, y = 0, y = \frac{1-x}{2}$.

V is the set $\left\{ (x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq \frac{1-x}{2}, 0 \leq z \leq 1-x-2y \right\}$, which is the solid enclosed by $x = 0, y = 0, z = 0, z = 1-x-2y$.

Now, suppose the projection of V onto y - z plane is σ_{yz} . Let $x = 0, z = 1-x-2y \Rightarrow z = 1-2y$ at $x=0$. Therefore,

σ_{yz} is the set $\left\{ (y, z) : 0 \leq y \leq \frac{1}{2}, 0 \leq z \leq 1-2y \right\}$.

V can also be described as the set $\left\{ (x, y, z) : 0 \leq y \leq \frac{1}{2}, 0 \leq z \leq 1-2y, 0 \leq x \leq 1-2y-z \right\}$.

$$\int_0^1 \left[\int_0^{\frac{1-x}{2}} \left(\int_0^{1-x-2y} f(x, y, z) dz \right) dy \right] dx = \int_0^{\frac{1}{2}} \left[\int_0^{1-2y} \left(\int_0^{1-2y-z} f(x, y, z) dx \right) dz \right] dy$$

9. Let V be the region in the first octant, where $x, y, z \geq 0$ bounded by $x^2 + y^2 = 1$, $x = 0, y = 0, z = 0, z = 1$. Using cylindrical polar coordinate, compute $\iiint_V xy dx dy dz$.

Solution:

The projection of V on xy -plane is $\begin{cases} x^2 + y^2 \leq 1 \\ x, y \geq 0 \end{cases}$. Then

$$\begin{aligned} \iiint_V xy dx dy dz &= \int_0^1 \left[\int_0^{\frac{\pi}{2}} \left(\int_0^1 r^3 \cos \theta \sin \theta dz \right) d\theta \right] dr = \int_0^1 \left[\int_0^{\frac{\pi}{2}} r^3 \cos \theta \sin \theta d\theta \right] dr = \int_0^1 \left(\frac{r^3}{2} \sin^2 \theta \right) \Big|_0^{\frac{\pi}{2}} d\theta \\ &= \int_0^1 \frac{r^3}{2} dr = \frac{r^4}{8} \Big|_0^1 = \frac{1}{8} \end{aligned}$$

10. In a sample model of the charge distribution around the positively charged (Q) nucleus of the hydrogen atom the charge density at the point (x, y, z) in the electron cloud is $f(x, y, z) = \frac{-Q}{\pi a^3} e^{-\frac{2\sqrt{x^2+y^2+z^2}}{a}}$

where a is the Bohr radius. Determine the total charge in the electron cloud.

Solution:

Under the transformations $\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases}$, R^3 is transformed into V' : $\begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \\ 0 \leq \rho < \infty \end{cases}$.

$$\begin{aligned} q &= \iiint_{R^3} \left(\frac{-Q}{\pi a^3} e^{-\frac{2\sqrt{x^2+y^2+z^2}}{a}} \right) dx dy dz = \iiint_{V'} \frac{-Q}{\pi a^3} e^{-\frac{2\rho}{a}} \rho^2 \sin \phi d\theta d\phi d\rho \\ &= \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \left(\int_0^\infty \frac{-Q}{\pi a^3} e^{-\frac{2\rho}{a}} \rho^2 \sin \phi d\rho \right) d\theta d\phi = \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \left[\int_0^\infty \frac{Q}{2\pi a^2} \sin \phi \rho^2 d\left(e^{-\frac{2\rho}{a}}\right) \right] d\theta d\phi \\ &= \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \frac{Q}{2\pi a^2} \sin \phi \left[\rho^2 e^{-\frac{2\rho}{a}} \Big|_0^\infty - \int_0^\infty e^{-\frac{2\rho}{a}} d(\rho^2) \right] d\theta d\phi \\ &= \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \frac{Q}{2\pi a^2} \sin \phi \left(-\int_0^\infty 2e^{-\frac{2\rho}{a}} \rho d\rho \right) d\theta d\phi = \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \frac{Q}{2\pi a} \sin \phi \left[\int_0^\infty \rho d\left(e^{-\frac{2\rho}{a}}\right) \right] d\theta d\phi \\ &= \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \frac{Q}{2\pi a} \sin \phi \left(\rho e^{-\frac{2\rho}{a}} \Big|_0^\infty - \int_0^\infty e^{-\frac{2\rho}{a}} d\rho \right) d\theta d\phi = \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \frac{Q}{2\pi a} \sin \phi \left(-\int_0^\infty e^{-\frac{2\rho}{a}} d\rho \right) d\theta d\phi \\ &= \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \frac{Q}{4\pi} \sin \phi \left(e^{-\frac{2\rho}{a}} \Big|_0^\infty \right) d\theta d\phi = \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} -\frac{Q}{4\pi} \sin \phi d\theta d\phi \\ &= \int_0^\pi \left(\int_0^{2\pi} \frac{-Q}{4\pi} \sin \phi d\theta \right) d\phi = \int_0^\pi \frac{-Q}{2} \sin \phi d\phi = \frac{-Q}{2} \cos \phi \Big|_0^\pi = -Q \end{aligned}$$

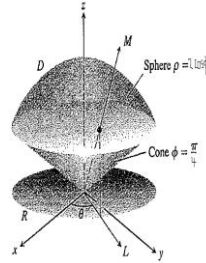
Note: To evaluate $\rho^2 e^{-\frac{2\rho}{a}} \Big|_0^\infty$, differentiate with respect to ρ twice the top and the bottom of $\frac{\rho^2}{e^{\frac{2\rho}{a}}}$

$$\rho^2 e^{-\frac{2\rho}{a}} \Big|_0^\infty = \lim_{\rho \rightarrow \infty} \frac{\rho^2}{e^{\frac{2\rho}{a}}} = \lim_{b \rightarrow \infty} \frac{2\rho}{\frac{2}{a} e^{\frac{2\rho}{a}}} = \lim_{b \rightarrow \infty} \frac{2}{\frac{4}{a^2} e^{\frac{2\rho}{a}}} = 0$$

11. (Optional) Let V be the region enclosed by both the surfaces: $\begin{cases} x^2 + y^2 + (z-1)^2 = 1 \\ z \geq 1 \end{cases}$ and $x^2 + y^2 = z^2$.

Using spherical polar coordinate, compute $\iiint_V z dx dy dz$.

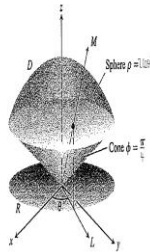
Solution:



We integrate over the solid enclosed by the upper hemisphere $\begin{cases} x^2 + y^2 + (z-1)^2 = 1 \\ z \geq 1 \end{cases}$ with center at $(0,0,1)$

and radius 1 and by the cone $x^2 + y^2 = z^2$.

See the figure below.



$$\begin{aligned} \iiint_V z dx dy dz &= \int_0^{\frac{\pi}{4}} \left(\int_0^{2\cos\phi} \left[\int_0^{2\pi} (\rho \cos\phi) (\rho^2 \sin\phi) d\theta \right] d\rho \right) d\phi = \int_0^{\frac{\pi}{4}} \left(\int_0^{2\cos\phi} 2\pi \rho^3 \sin\phi \cos\phi d\rho \right) d\phi \\ &= \int_0^{\frac{\pi}{4}} \left(\frac{\pi \rho^4}{2} \sin\phi \cos\phi \right) \Big|_0^{2\cos\phi} d\phi = \int_0^{\frac{\pi}{4}} 8\pi \sin\phi \cos^5\phi d\phi = - \int_0^{\frac{\pi}{4}} 8\pi \cos^5\phi d(\cos\phi) \\ &= - \frac{4\pi \cos^6\phi}{3} \Big|_0^{\frac{\pi}{4}} = - \frac{4\pi}{3} \left[\left(\frac{1}{\sqrt{2}} \right)^6 - 1 \right] = - \frac{4\pi \left(\frac{1}{8} - 1 \right)}{3} = \frac{7\pi}{6} \end{aligned}$$

-End-