

CITY UNIVERSITY OF HONG KONG,

EE3210 Signals & Systems

Lecture note

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Semester A, 2021/22

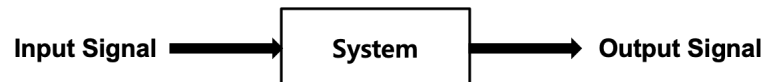
1 Signals and Systems

Major References:

- Chapter 1, *Signals and Systems* by Alan V. Oppenheim et. al., 2nd edition, Prentice Hall
- Chapter 1, *Schaum's Outline of Signals and Systems*, 2nd Edition, 2010, McGraw-Hill

Introduction

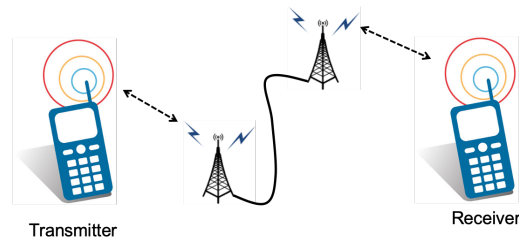
- **Objective:** Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



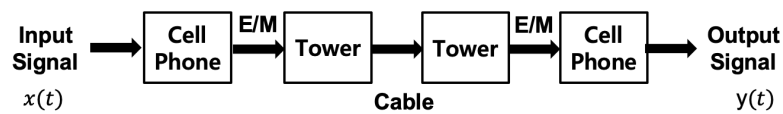
- **System level approach**
 - **Abstraction**
 - * Identify the system input and output signals → characterize the signal types
 - * Write input-output relation of the system → Operational transformations (e.g., Fourier analysis, Laplace Transformation).
 - * Characterize the system types by the input-output relation → system types
 - **Modular design**
 - * Break down the system into a number of interconnected subsystems (**module**)
 - * Each module performs some specific task.
 - * Focus on the **flow of signal**, abstract everything else away
 - **Composite system**
 - * Determine the input-output relationship between each modules
 - * Combine the components (**module**) to composite the overall system
 - * **Component** and **composite systems** have the same form, and are analyzed with same methods.

Example 1.1

Let's consider a typical mobile communication between the transmitter and receiver. Abstract the system input and output signals, then determine the input-output relationship.



Sol) We first describe each module as a cascade of component systems.

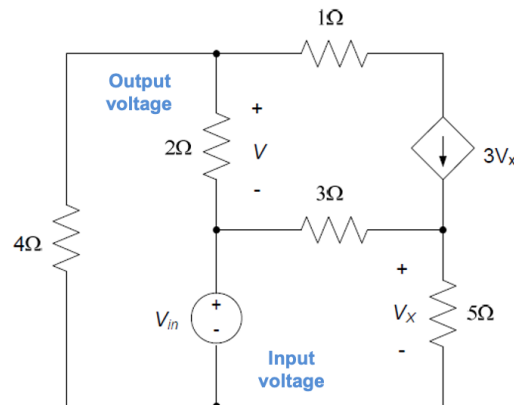


Then we combine the modules into a composite system.

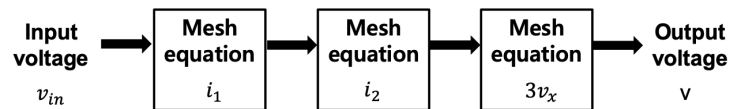
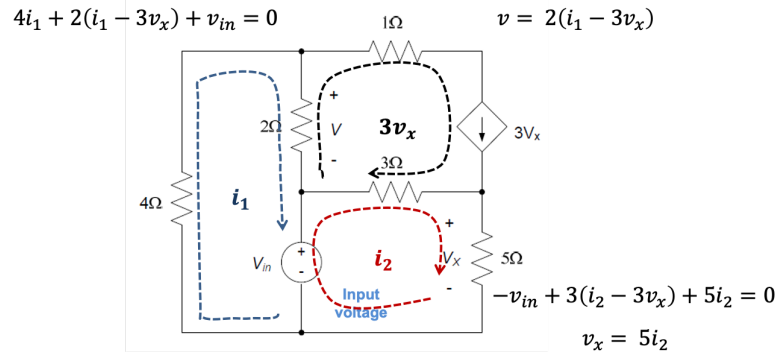


Example 1.2

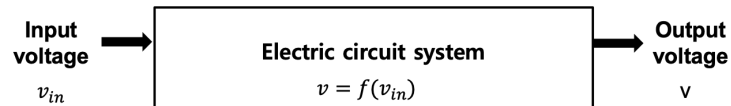
Determine the input-output relationship of the following electrical circuit.



Sol) We first characterize the input and output signals, then perform modular design by breaking the circuit into several modular circuits



Then we combine the modules into a composite system



1.1 Classification of Signal Types

- [Def] **Signal** $x(t)$ is a function of an independent variable t representing a physical behavior of the phenomenon.

1. Continuous-Time (CT) and Discrete-Time (DT) Signals

A signal $x(t)$ is a **continuous-time signal** if t is a continuous variable. If t is a discrete variable, i.e., $x(t)$ is defined at discrete times, then $x(t)$ is a **discrete-time signal**.

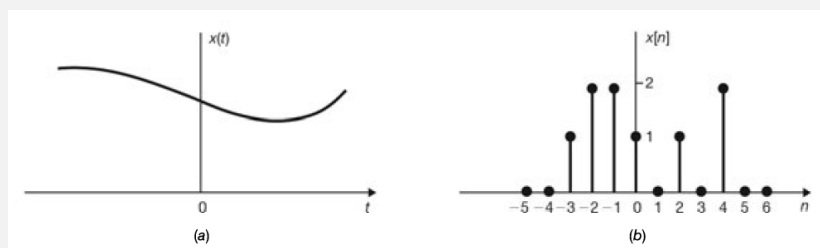


Fig. 1 (a) Continuous-time and (b) discrete-time signals

DT signal $x[n]$ may be obtained by *sampling* a CT signal $x(t)$ such as

$$x_n = x[n] = x(t_n)$$

Two representations of the DT signals

- Functional form. For example,

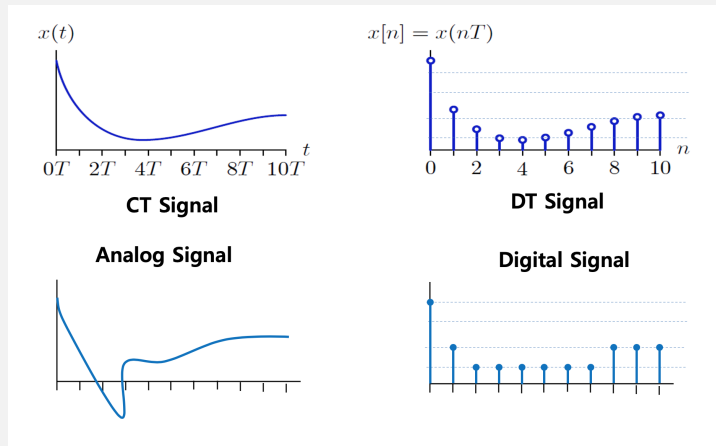
$$x_n = x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

- Sequence form. For instance,

$$\{x_n\} = \{\dots, 0, 0, 1, 2, 2, 1, 0, 1, 0, 2, 0, 0, \dots\}$$

2. Analog and Digital Signals

A CT signal $x(t)$ is an **analog signal** if $x(t)$ can take on any value in the continuous interval (a, b) . A DT signal $x[n]$ is a **digital signal** if $x[n]$ can take on only a finite number of distinct values.



3. Periodic and Aperiodic (or nonperiodic) Signals

A signal $x(t)$ (or $x[n]$) is **periodic** with period T (or N) if there is a positive non-zero value of T (or N) for which the following equality holds

$$\begin{aligned} x(t + T) &= x(t), & \text{for CT signal } x(t), \\ x[n + N] &= x[n], & \text{for DT signal } x[n] \end{aligned} \quad (1.1)$$

Any signal that is not periodic is called a **nonperiodic** (or **aperiodic**) signal.

- Property 1. For a given periodic signal, $\{T, 2T, 3T, \dots, mT, \dots\}$ are all available period at all t and any integer m .
- Property 2. **Fundamental period** T_0 is the smallest positive value of the period T for which (1.1) holds.

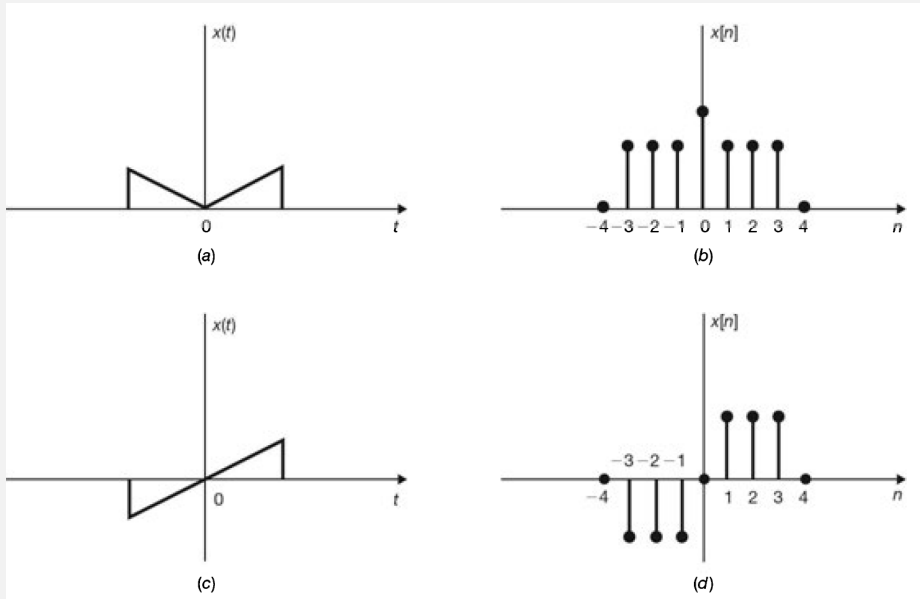
4. Even and Odd Signals

A signal is referred to as an **even signal** if

$$x(-t) = x(t) \quad \text{for CT signal } x(t), \quad x[-n] = x[n] \quad \text{for DT signal } x[n] \quad (1.2)$$

A signal is referred to as an **odd signal** if

$$x(-t) = -x(t) \quad \text{for CT signal } x(t), \quad x[-n] = -x[n] \quad \text{for DT signal } x[n] \quad (1.3)$$



- Property 1. Any signal $x(t)$ or $x[n]$ can be expressed as a sum of two signals

$$\begin{aligned} x(t) &= x_e(t) + x_o(t) \quad \text{for CT signal,} \\ x[n] &= x_e[n] + x_o[n] \quad \text{for DT signal,} \end{aligned} \quad (1.4)$$

where $x_e(t)$ (or $x_e[n]$) is the even part, $x_o(t)$ (or $x_o[n]$) is the odd part, and these components are related to the original signal $x(t)$ (or $x[n]$) as follows

$$\begin{aligned} x_e(t) &= \frac{1}{2} [x(t) + x(-t)], \\ x_o(t) &= \frac{1}{2} [x(t) - x(-t)]. \end{aligned} \quad (1.5)$$

- Property 2. Product of signals
 - Even signal $\times$ Even signal = Even signal, Odd $\times$ odd = Even signal
 - Even signal $\times$ odd signal = odd signal

5. Energy and Power Signals

Energy E of a signal $x(t)$ (or $x[n]$) is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad \text{for CT signal,} \quad E = \sum_{n=-\infty}^{\infty} |x[n]|^2 \quad \text{for DT signal,} \quad (1.6)$$

whereas the Power P of a signal is defined as follows

$$P = \begin{cases} \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt & \text{for CT signal,} \\ \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2 & \text{for DT signal} \end{cases} \quad (1.7)$$

- **Energy signal** has finite energy and zero power

$$0 < E < \infty, \quad P = 0.$$

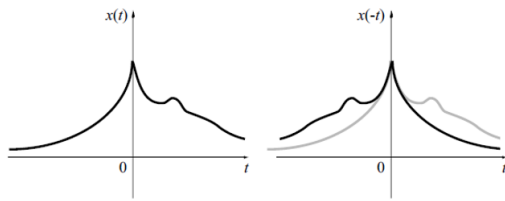
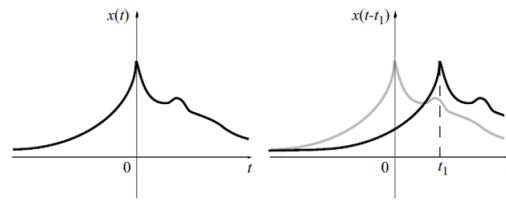
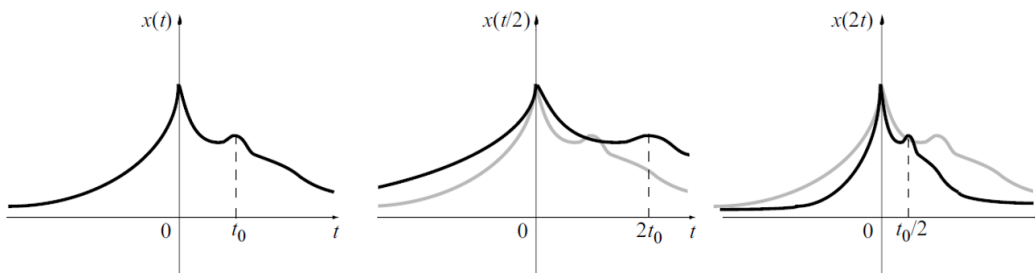
- **Power signal** has finite power and infinite energy

$$0 < P < \infty, \quad E = \infty.$$

- Signals that satisfy neither property are neither energy signals nor power signals.

1.2 Basic Signal Operations

- **Time Reversal:** Flip the signal around the vertical axis $x(t) \rightarrow x(-t)$
- **Time Shifts:** Shift the signal to left or right $x(t) \rightarrow x(t - t_0)$
 - **Right-shift** if $t_0 > 0$, **Left-shift** if $t_0 < 0$.
- **Time Scaling:** Linearly stretch or compress the signal $x(t) \rightarrow x(ct)$
 - **Compression** if $|c| > 1$, **Expansion** if $|c| < 1$.
- **Affine Transformation:** $x(t) \rightarrow x(\alpha t + \beta) = x(\alpha(t + \beta/\alpha))$ for any real α, β
 - Step 1. **Scale** by α . If $\alpha < 0$, reflection across y -axis
 - Step 2. **Shift** by $-\beta/\alpha$.
 - * If α and β have different signs, right-shift.
 - * If α and β have same signs, left shift.

Time Reflection: $x(t) \longrightarrow x(-t)$ **Time shifts:** $x(t) \longrightarrow x(t - t_1)$ **Time scaling:** $x(t) \longrightarrow x(ct)$ 

1.3 Example of Important Signals

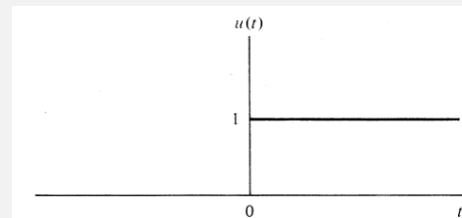
1. Unit Step Function (also referred as Heaviside unit function)

- **Definition**

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0, \end{cases} \quad (1.8)$$

- **Properties**

- Aperiodic signal
- Power signal $P = 1/2$
- Infinite Energy $E = \infty$



Functions related to the step function $u(t)$

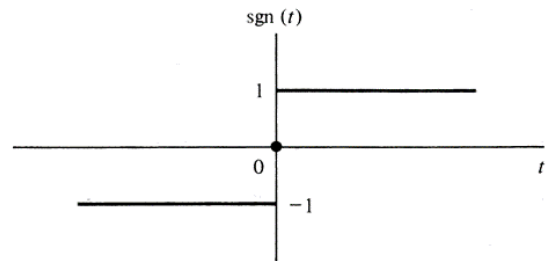
a) Signum Function

- **Definition**

$$\text{sgn}(t) = 2u(t) - 1 = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

- Properties

- Aperiodic & odd signal
- Power signal $P = 1$
- Infinite Energy $E = \infty$



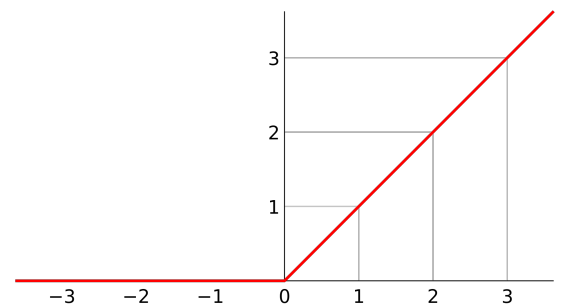
b) **Ramp Function**

- Definition

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}, \quad \int_{-\infty}^t u(\tau) d\tau = r(t)$$

- Properties

- Aperiodic
- Infinite Power $P = \infty$
- Infinite Energy $E = \infty$



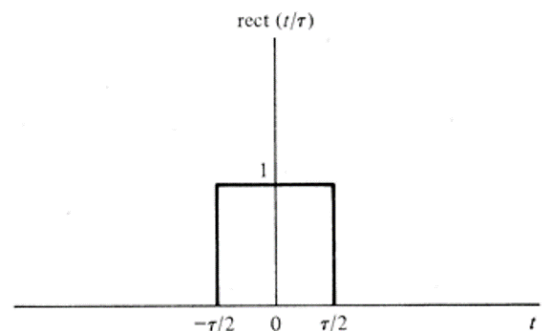
c) **Rectangular Pulse**

- Definition

$$\text{rect}(t/\tau) = \begin{cases} 1, & |t| < \frac{\tau}{2} \\ 0, & |t| > \frac{\tau}{2} \end{cases}, \quad \text{rect}(t/\tau) = u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right) \quad (1.9)$$

- Properties

- Aperiodic & Even signal
- Zero Power $P = 0$
- Energy Signal $E = \tau$



2. Unit Impulse Function (also referred as *Dirac delta function*)

• Definition

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (1.10)$$

• Properties

– Sampling Property

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

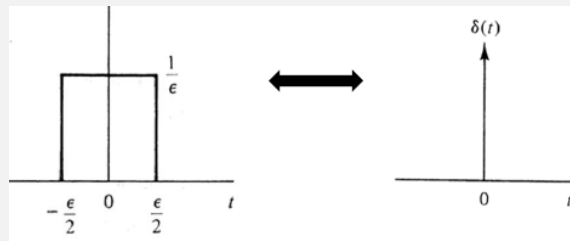
– Sifting Property

$$\int_a^b x(t)\delta(t - t_0) dt = \begin{cases} x(t_0), & \text{if } a < t_0 < b \\ 0, & \text{otherwise} \end{cases}$$

- Impulse function is the *building block of any signal*, i.e., arbitrary signal can be represented as an infinite sum of impulse function and signal amplitude.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau) d\tau \quad (1.11)$$

Relationship between Rectangular Pulse and Impulse Function



- | | |
|--|--|
| • $\delta_\epsilon(t) = \frac{1}{\epsilon} \text{rect}\left(\frac{t}{\epsilon}\right)$ | • $\delta(t) = \lim_{\epsilon \rightarrow 0} \delta_\epsilon(t)$ |
| • $\delta_\epsilon(0) = \frac{1}{\epsilon}$ | • $\delta(0) \rightarrow \infty$ |
| • $\delta_\epsilon(t) = 0, t > \frac{\epsilon}{2}$ | • $\delta(t) = 0, t \neq 0$ |
| • $\int_{-\infty}^{\infty} \delta_\epsilon(t) dt = 1$ | • $\int_{-\infty}^{\infty} \delta(t) dt = 1$ |

Additional Properties of Unit impulse function

- Scaling Property:

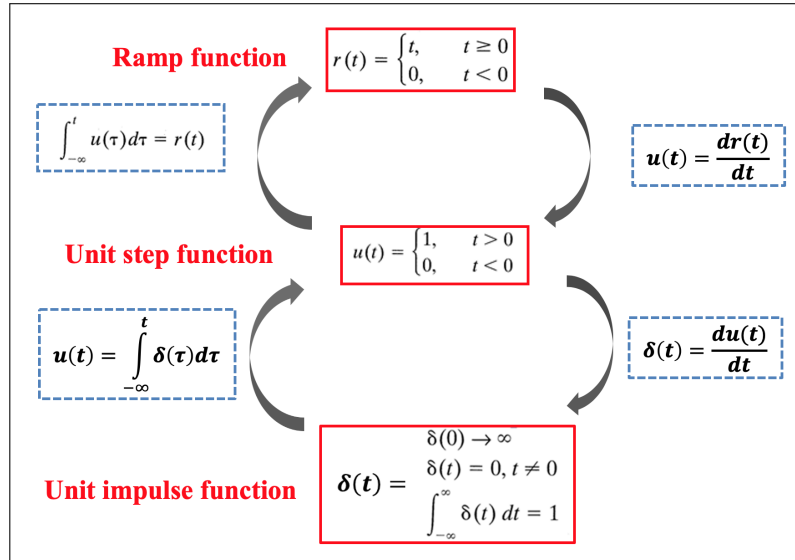
$$\delta(at) = \frac{1}{|a|} \delta(t)$$

- Even Function:

$$\delta(-t) = \delta(t)$$

- Derivative and Integral:

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau, \quad \delta(t) = \frac{du(t)}{dt}$$

**3. Complex Exponential Function**

- **Definition**

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j \sin(\omega_0 t)$$

- **Properties**

- Periodic with $T = \frac{2\pi n}{\omega_0}$ where n is an integer

- Fundamental period $T_0 = \frac{2\pi}{|\omega_0|}$

- Infinite Energy $E = \infty$

- Finite power $P = \lim_{T_0 \rightarrow \infty} \frac{1}{T_0} \int_0^{T_0} |e^{j\omega_0 t}|^2 dt = \lim_{T_0 \rightarrow \infty} \frac{1}{T_0} \int_0^{T_0} 1 \cdot dt = 1$

4. Sinusoidal Function

$$A \cos(\omega_0 t + \theta) \quad \text{or} \quad A \sin(\omega_0 t + \theta),$$

where A is the *amplitude*, θ is the *phase angle*, ω_0 is the *radian frequency* with

$$\text{Fundamental period } T_0 = \frac{2\pi}{\omega_0} \text{ (sec),} \quad \text{Fundamental frequency } f_0 = \frac{1}{T_0} \text{ hertz (Hz)}$$

1.4 Classification of System Types

- **[Def]** A *system* is a mathematical model of a physical process that relates the *input signal* to the *output signal* in the form $y = Tx$.

1. Invertible and Noninvertible System

A system is said to be **invertible** if distinct inputs lead to distinct outputs. Otherwise, the system is said to be **noninvertible**.

[Examples]

Invertible System

- $y(t) = 2x(t) \leftrightarrow w(t) = \frac{1}{2}y(t)$
- $y[n] = \sum_{k=-\infty}^n x[k] \leftrightarrow w[n] = y[n] - y[n-1]$

Noninvertible System

- $y[n] = 0$
- $y(t) = x^2(t)$

2. Memory and Memoryless System

A system is said to be **memoryless** if the output at any time depends only on the input at that same time. Otherwise, the system is said to have **memory**.

[Examples]

Memoryless System

- $y(t) = Rx(t)$
- $y[n] = (2x[n] - x^2[n])^2$
-

System with Memory

- $y[n] = \sum_{k=-\infty}^n x[k]$
- $y[n] = x[n-1]$
- $y(t) = \frac{1}{c} \int_{-\infty}^t x(\tau) d\tau$

3. Causal and Noncausal System

A system is said to be **causal** if its output at the present time depends on only the present and/or past values of the input. If its output at the present time depends on future values of the input, the system is known as **noncausal**.

[Examples]

Causal System

- $y[n] = \sum_{k=-\infty}^n x[k]$
- $y(t) = x^2(t)$

Noncausal System

- $y[n] = x[n] + x[n+2]$
- $y[n] = x[-n]$ or $y(t) = x(t+1)$

* **Note)** All memoryless systems are causal, but not vice versa.

4. Linear and Nonlinear System

A system is said to be **linear** if the following superposition property (1.12) holds for a given operator T . If the system does not satisfy (1.12), it is a **nonlinear system**.

$$T\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 T\{x_1\} + \alpha_2 T\{x_2\} \quad (1.12)$$

[Examples]**Linear System**

- $y[n] = \sum_{k=-\infty}^n x[k]$
- $y(t) = tx(t)$

Nonlinear System

- $y(t) = x^2(t)$
- $y[n] = 2x[n] + 3$

* **Note)** For a linear system, zero input always yields a zero output.

5. Time-invariant and Time-Varying System

A system is **time-invariant** if a time-shift of the input causes a corresponding shift in the output. In other words, the system response is independent of time.

$$\text{If } y(t) = T\{x(t)\}, \text{ then } y(t - t_0) = T\{x(t - t_0)\} \quad (1.13)$$

[Examples]**Time invariant System**

- $y[n] = \sum_{k=-\infty}^n x[k]$
- $y[n] = x[n - n_0]$

Time varying System

- $y(t) = x(2t)$
- $y[n] = nx[n]$

LTI System

Linear time-invariant (LTI) system: A system that is linear and also time-invariant.

6. Stable and Unstable System

A system is **stable** if every bounded input produces a bounded output for all time.

$$\text{If } |x(t)| < A, \text{ then } |y(t)| < B \text{ where } |A| < \infty, |B| < \infty \quad (1.14)$$

[Examples]**Stable System**

- $y(t) = x^2(t)$
- $y[n] = x[n] + x[n + 2]$

Unstable System

- $y[n] = \frac{1}{x[n]}$
- $y[n] = nx[n]$

1.5 Examples

[Example 1-1] Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

- | | |
|---|--|
| a) $x(t) = \cos\left(t + \frac{\pi}{4}\right)$ | b) $x(t) = \sin\left(\frac{2\pi t}{3}\right)$ |
| c) $x(t) = \cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{4}\right)$ | d) $x(t) = \cos(t) + \sin(\sqrt{2}t)$ |
| e) $x(t) = \sin^2(t)$ | f) $x(t) = e^{j\left[\frac{\pi}{2}t - 1\right]}$ |
| g) $x(t) = \cos\left(2t + \frac{\pi}{4}\right)$ | h) $x(t) = \cos^2(t)$ |
| i) $x(t) = (\cos(2\pi t))u(t)$ | j) $x(t) = e^{j\pi t}$ |

Solution) To solve this type of problem, try to find the minimum T that satisfy $x(t+T) = x(t)$. For instance, in (a), if the following equality holds with a nonzero constant T , then it is periodic

$$\cos\left(t + \frac{\pi}{4}\right) = \cos\left(t + T + \frac{\pi}{4}\right) \rightarrow \cos(t') = \cos(t' + T), \quad (1.15)$$

where we used a *change of variable* $t' = t + \frac{\pi}{4}$ in the second equality. Since the minimum T that satisfy (1.15) is 2π , (a) is a periodic signal with period $T = 2\pi$. Similarly, for (b),

$$\sin\left(\frac{2\pi t}{3}\right) = \sin\left(\frac{2\pi t}{3} + \frac{2\pi T}{3}\right) \rightarrow \frac{2\pi T}{3} = 2\pi, \quad (1.16)$$

and by denoting $t' = \frac{2\pi t}{3}$, the minimum T that satisfy (1.16) is 3.

For (c) and (d), we can use (1.18); The period T_1 for $\cos\left(\frac{\pi t}{3}\right)$ in (c) is $T_1 = 6$ and T_2 for $\sin\left(\frac{\pi t}{4}\right)$ is $T_2 = 8$. Since $T_1/T_2 = 3/4$, (c) is a periodic signal with period $T = 24$. In (d), the period T_1 for $\cos(t)$ is $T_1 = 2\pi$ and T_2 for $\sin(\sqrt{2}t)$ is $T_2 = \sqrt{2}\pi$. Since $T_1/T_2 = \sqrt{2}$, (d) is aperiodic signal.

For (e) and (h), convert $x(t)$ as follows, then apply similar approach as (a).

$$\cos^2(t) = \frac{1}{2}(1 + \cos(2t)), \quad \sin^2(t) = \frac{1}{2}(1 - \cos(2t)), \quad (1.17)$$

and the remaining can be solved using similar method. The solutions are summarized below.

- | | | |
|-----------------------------|----------------------------|---------------------------|
| a) Periodic with $T = 2\pi$ | b) Periodic with $T = 3$ | c) Periodic with $T = 24$ |
| d) Aperiodic | e) Periodic with $T = \pi$ | f) Periodic with $T = 4$ |
| g) Periodic with $T = \pi$ | h) Periodic with $T = \pi$ | i) Aperiodic |
| j) Periodic with $T = 2$ | | |

Sum of Periodic Signals

- Let $x_1(t)$ and $x_2(t)$ be periodic signals with fundamental periods T_1 and T_2 , respectively. The sum $x(t) = x_1(t) + x_2(t)$ is periodic if and only if the following condition holds

$$\frac{T_1}{T_2} = \frac{k}{m} = \text{rational number} \quad (1.18)$$

where the fundamental period T is the least common multiple of T_1 and T_2 .

- Let $x_1[n]$ and $x_2[n]$ be periodic sequence with fundamental periods N_1 and N_2 , respectively. The sum $x[n] = x_1[n] + x_2[n]$ is periodic given the following condition

$$mN_1 = kN_2 = N \quad (1.19)$$

where the fundamental period N is the least common multiple of N_1 and N_2 .

Refer [Schaum's text, Problem 1.14 & 1.15]



[Example 1-2] Determine whether the following signals are energy signals, power signals, or neither.

a) $x(t) = e^{-at}u(t)$, $a > 0$

b) $x(t) = A \cos(\omega_0 t + \theta)$

Solution) To solve this type of problem, **(Step 1.)** you need to calculate the energy E first. If E is finite, the signal is a Energy signal. Otherwise, **(Step 2.)** if E is infinite, you need to calculate the power P as well. If P is finite, the signal is a Power signal. Otherwise, if P is infinite, then it is neither a energy nor a power signal. For example, in (a),

$$E = \int_{-\infty}^{\infty} e^{-2at}u(t)dt = \int_0^{\infty} e^{-2at}dt = \frac{1}{2a}, \quad (1.20)$$

where we used the definition of the step function in the second equality. Since $\frac{1}{2a}$ is finite, $x(t)$ in (a) is a energy signal. For a periodic signal, the integration interval T in (1.7) is equal to the period. In (b), the period is $T = \frac{2\pi}{\omega_0}$ and the signal power can be calculated as follows

$$\begin{aligned} P &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T A^2 \cos^2(\omega_0 t + \theta) dt = \lim_{T \rightarrow \infty} \frac{A^2}{2\pi} \int_{\theta}^{2\pi+\theta} \cos^2(l) dl \\ &= \lim_{T \rightarrow \infty} \frac{A^2}{4\pi} \int_{\theta}^{2\pi+\theta} [1 + \cos(2l)] dl = \frac{A^2}{2}, \end{aligned} \quad (1.21)$$

where we used $T = \frac{2\pi}{\omega_0}$ and a change of variable, $l = \omega_0 t + \theta$ or $\omega_0 dt = dl$, in the second equality, then applied the Cosine rule $\cos^2(t) = \frac{1}{2}(1 + \cos(2t))$ in the third equality. Since $\frac{A^2}{2}$ is finite, $x(t)$ in (b) is a power signal. In summary, the solutions are

a) Energy signal

b) Power signal

Definition of Energy and Power Signals

- **Energy signal** has finite energy and zero power, i.e., $0 < E < \infty$, $P = 0$
- **Power signal** has finite power and infinite energy, i.e., $0 < P < \infty$, $E = \infty$, where

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt, \quad P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

Properties of Periodic Signals

The following equalities hold for a periodic signal $x(t + T) = x(t)$

$$\int_{\alpha}^{\beta} x(t) dt = \int_{\alpha+T}^{\beta+T} x(t) dt, \quad \int_0^T x(t) dt = \int_a^{a+T} x(t) dt,$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt,$$

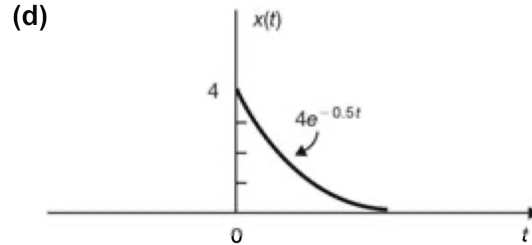
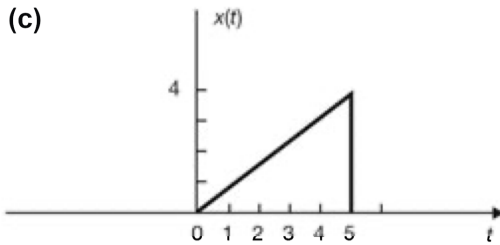
where T_0 is the fundamental period and α, β, a are arbitrary real valued constants.
Refer [Schaum's text, Problem 1.17 & 1.18]



[Example 1-3] Determine the even and odd component of the following signals

a) $x(t) = u(t)$

b) $x(t) = \sin(\omega_0 t + \frac{\pi}{4})$



Solution) To solve this type of problem, you need to apply (1.22). In (a), $x(-t) = u(-t) = 1$ for $t < 0$ and $u(-t) = 0$ for $t > 0$. Then, the following results can be derived

$$x_e(t) = \frac{1}{2} [u(t) + u(-t)] = \frac{1}{2},$$

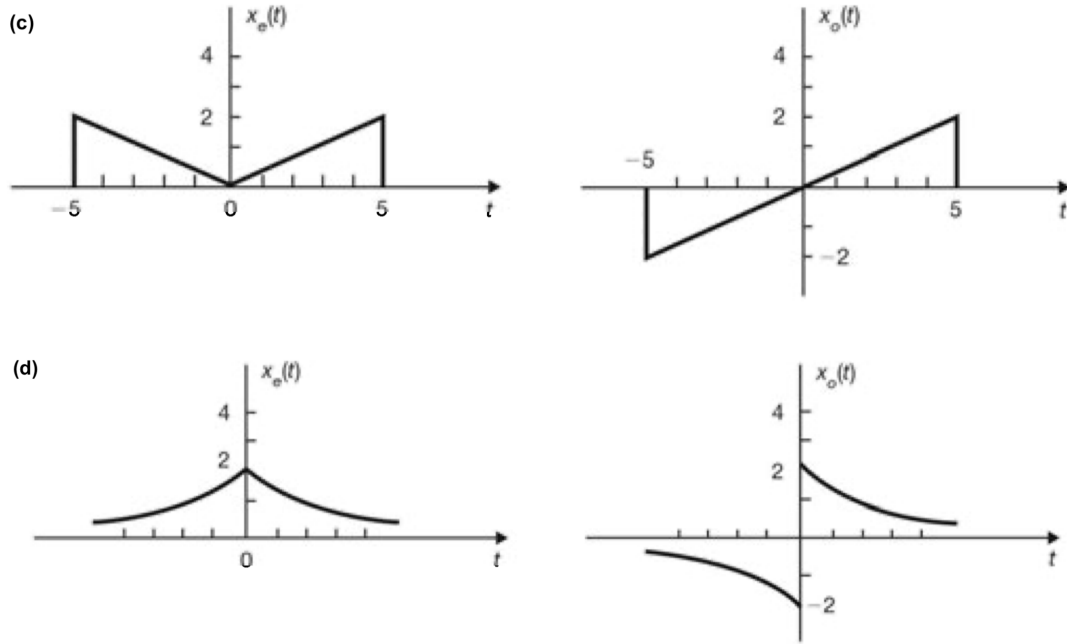
$$x_o(t) = \frac{1}{2} [u(t) - u(-t)] = \frac{1}{2} \text{sgn}(t) = \begin{cases} 0.5, & t > 0, \\ -0.5, & t < 0 \end{cases}$$

In (b), we first use Sine rule to expand the Sine function, then the following results can be derived.

$$\sin\left(\omega_0 t + \frac{\pi}{4}\right) = \sin(\omega_0 t) \cos\left(\frac{\pi}{4}\right) + \cos(\omega_0 t) \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} (\sin(\omega_0 t) + \cos(\omega_0 t)).$$

$$x_e(t) = \frac{1}{\sqrt{2}} \cos(\omega_0 t), \quad x_o(t) = \frac{1}{\sqrt{2}} \sin(\omega_0 t).$$

Similarly, the even and odd component of (c) and (d) can be found as follows



Even and Odd Component

Any signal $x(t)$ can be expressed as a sum of two signals

$$x(t) = x_e(t) + x_o(t),$$

where $x_e(t)$ and $x_o(t)$ are related to the original signal $x(t)$ as follows

$$x_e(t) = \frac{1}{2} [x(t) + x(-t)], \quad x_o(t) = \frac{1}{2} [x(t) - x(-t)]. \quad (1.22)$$



[Example 1-4] [Part 1] Sketch the following signals.

a) $x_1(t) = u(t) + 5u(t-1) - 2u(t-2)$

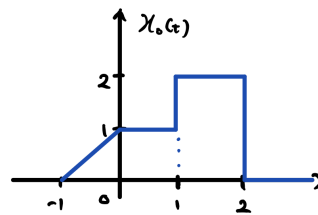
b) $x_2(t) = r(t) - r(t-1) - u(t-2)$

c) $x_3(t) = u(t)u(a-t), a > 0$

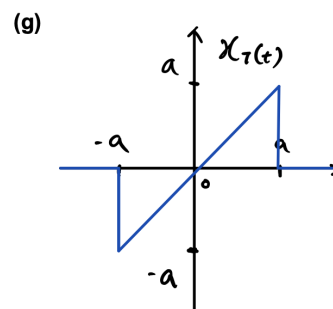
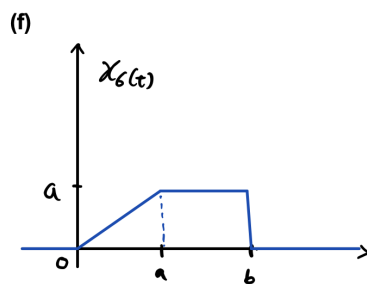
d) $x_4(t) = x_0(t)u(1-t)$

e) $x_5(t) = x_0(t) [u(t) - u(t - 1)]$

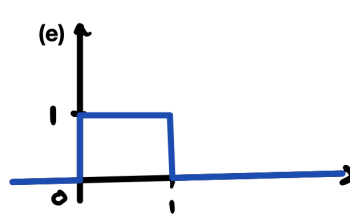
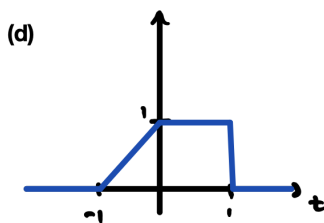
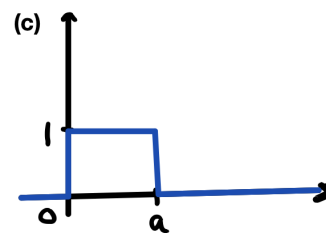
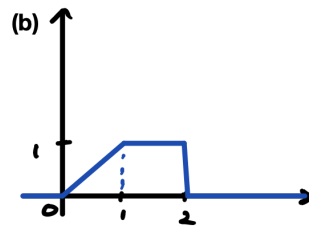
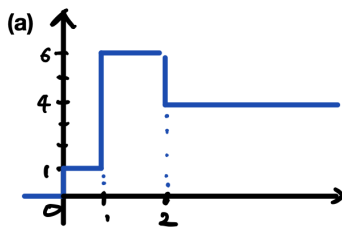
where the signal $x_0(t)$ is plotted below.



[Part 2] For each of the signals plotted below, write an expression in terms of unit step and unit ramp functions.



Solution)[Part 1]



[Part 2]

(f) $x_6(t) = r(t) - r(t - a) - au(t - b)$, (g) $x_7(t) = (r(t) - r(-t))(u(t + a) - u(t - a))$

- **Time Reversal:** Flip the signal around the vertical axis $x(t) \rightarrow x(-t)$
- **Time Shifts:** Shift the signal to left or right $x(t) \rightarrow x(t - t_0)$
 - **Right-shift** if $t_0 > 0$, **Left-shift** if $t_0 < 0$.
- **Time Scaling:** Linearly stretch or compress the signal $x(t) \rightarrow x(ct)$
 - **Compression** if $|c| > 1$, **Expansion** if $|c| < 1$.



[Example 1-5] Evaluate the following integrals.

- | | |
|---|---|
| a) $\int_{-\infty}^t \cos(\tau) u(\tau) d\tau$ | b) $\int_{-\infty}^t \cos(\tau) \delta(\tau) d\tau$ |
| c) $\int_{-\infty}^{\infty} \cos(t) u(t-1) \delta(t) dt$ | d) $\int_0^{2\pi} t \sin\left(\frac{t}{2}\right) \delta(t-\pi) dt$ |
| e) $\int_{-\infty}^{\infty} \left(\frac{2}{3}t - \frac{3}{2}\right) \delta(t-1) dt$ | f) $\int_{-3}^2 \left[\exp(1-t) + \sin\left(\frac{2\pi t}{3}\right)\right] \delta\left(t - \frac{3}{2}\right) dt$ |

Solution)

(a)

$$\int_{-\infty}^t \cos(\tau) u(\tau) d\tau = \begin{cases} \text{If } t > 0, \int_0^t \cos(\tau) d\tau = \sin(t) \\ \text{If } t < 0, 0 \end{cases} = u(t) \sin(t)$$

(b)

$$\int_{-\infty}^t \cos(\tau) \delta(\tau) d\tau = \begin{cases} \text{If } t > 0, \cos 0 = 1 \\ \text{If } t < 0, 0 \end{cases} = u(t)$$

(c)

$$\int_{-\infty}^{\infty} \cos(t) u(t-1) \delta(t) dt = \cos(0) u(-1) = 0$$

(d)

$$\int_0^{2\pi} t \sin\left(\frac{t}{2}\right) \delta(t-\pi) dt = \pi \sin\left(\frac{\pi}{2}\right) = \pi$$

(e)

$$\int_{-\infty}^{\infty} \left(\frac{2}{3}t - \frac{3}{2}\right) \delta(t-1) dt = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}$$

(f)

$$\int_{-3}^2 \left[\exp(1-t) + \sin\left(\frac{2\pi t}{3}\right)\right] \delta\left(t - \frac{3}{2}\right) dt = \exp\left(-\frac{1}{2}\right) + \sin(\pi) = \exp(-0.5)$$

Properties of Unit impulse function

- $\int_a^b x(t) \delta(t - t_0) dt = \begin{cases} x(t_0), & \text{if } a < t_0 < b \\ 0, & \text{otherwise} \end{cases}$
- $x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$
- $\delta(at) = \frac{1}{|a|} \delta(t)$, $u(t) = \int_{-\infty}^t \delta(\tau) d\tau$, $\delta(t) = \frac{du(t)}{dt}$



[Example 1-6] Determine whether the following system is (i) memoryless, (ii) causal, (iii) linear, (iv) time-invariant, or (v) stable. Refer [Schaum's text, Problem 1.33, 1.34, 1.36, 1.38]

- a) $y(t) = \frac{1}{C} \int_{-\infty}^t x(\tau) d\tau$ b) $y(t) = x(t) \cos(\omega_0 t)$
 c) $y[n] = x[n - 1]$ d) $y[n] = nx[n]$

Solution In (a), the output depends on the past input, so it is not memoryless system. The output depends on the present and past values of the input, so it is a Causal system. To test linearity, substitute $x(t) \leftarrow \alpha_1 x_1(t) + \alpha_2 x_2(t)$ as the input, where $y_1(t)$ and $y_2(t)$ is the corresponding output of $x_1(t)$ and $x_2(t)$, respectively. Then,

$$\begin{aligned} y(t) &= \frac{1}{C} \int_{-\infty}^t [\alpha_1 x_1(\tau) + \alpha_2 x_2(\tau)] d\tau \\ &= \alpha_1 \left[\frac{1}{C} \int_{-\infty}^t x_1(\tau) d\tau \right] + \alpha_2 \left[\frac{1}{C} \int_{-\infty}^t x_2(\tau) d\tau \right] = \alpha_1 y_1(t) + \alpha_2 y_2(t), \end{aligned}$$

so the superposition property holds, which indicates a linear system. To test time-invariance, input time shifted signal $x(t - t_0)$. If the corresponding output is $y(t - t_0)$, then it is a time invariant system.

$$\frac{1}{C} \int_{-\infty}^t x(\tau - t_0) d\tau = \frac{1}{C} \int_{-\infty}^{t-t_0} x(l) dl = y(t - t_0),$$

by using a change of variable $l = \tau - t_0$ in the first equality. Hence, it is a time-invariant system. For stability, (a) can be easily proved to be a unstable by substituting a unit step function $x(t) = u(t)$ as the input, which achieves unbounded $y(t) = \frac{tu(t)}{C}$. The remaining can be proved using similar method. The solutions are summarized below.

- a) memory, causal, linear, time-invariant, unstable b) memoryless, causal, linear, time-variant, stable
 c) memory, causal, linear, time-invariant, stable d) memoryless, causal, linear, time-variant, unstable.

System Characterization

1. **Memoryless System**; output at any time depends only on the input at that same time
2. **Causal System**; output at the present time depends only on the present and/or past input values
3. **Linear System**; the superposition property holds, i.e., $T\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 T\{x_1\} + \alpha_2 T\{x_2\}$
4. **Time-invariant System**; time-shift of the input causes a same amount of shifting in the output
5. **Stable System**; If $|x(t)| < A$, then $|y(t)| < B$ where $|A| < \infty$, $|B| < \infty$
6. **LTI System**; A system that is linear and also time-invariant



2 Linear Time-Invariant Systems

Major References:

- Chapter 2, *Signals and Systems* by Alan V. Oppenheim et. al., 2nd edition, Prentice Hall
- Chapter 2, *Schaum's Outline of Signals and Systems*, 2nd Edition, 2010, McGraw-Hill

2.1 Convolution

2.1.1 Convolution Integral of CT Signal

1. Definition

Convolution Integral of two continuous-time signals $x(t)$ and $y(t)$ is defined by

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau. \quad (2.1)$$

Convolution $x(t) * y(t)$ represents the degree to which x & y overlap at t as y sweeps across the domain t .

- Step. 1) $y(\tau)$ is time-reversed, then shifted by t ; $y(\tau) \rightarrow y(-\tau) \rightarrow y(t - \tau)$
 Step. 2) $x(\tau)$ and $y(t - \tau)$ are multiplied, then integrated over τ
 Step. 3) Convolution will remain zero as long as x & y do not overlap
 Step. 4) Sweep $y(t - \tau)$ from $t = -\infty$ to $t = \infty$ to produce the entire output

2. Properties of the Convolution Integral

The convolution integral has the following properties. Refer [Schaum's text, Problem 2.1] for the proof.

a) Commutative

$$x(t) * y(t) = y(t) * x(t)$$

b) Associative

$$\{x(t) * y_1(t)\} * y_2(t) = x(t) * \{y_1(t) * y_2(t)\}$$

c) Distributive

$$x(t) * \{y_1(t) + y_2(t)\} = x(t) * y_1(t) + x(t) * y_2(t)$$

3. Additional Properties

Refer [Schaum's text, Problem 2.2, 2.8] for the proof.

- a) $x(t) * \delta(t) = x(t)$
 b) $x(t) * \delta(t - t_0) = x(t - t_0)$
 c) $x(t) * u(t) = \int_{-\infty}^t x(\tau) d\tau$
 d) $x(t) * u(t - t_0) = \int_{-\infty}^{t-t_0} x(\tau) d\tau$
 e) If $x(t)$ and $y(t)$ are periodic signals with a common period T , the convolution in (2.1) does not converge. Instead, we define the *periodic convolution* $f(t) = x(t) \otimes y(t)$, where $f(t)$ is periodic with period T .

$$\begin{aligned} f(t) &= x(t) \otimes y(t) = \int_0^T x(\tau) y(t - \tau) d\tau \\ &= \int_a^{a+T} x(\tau) y(t - \tau) d\tau \quad \text{for arbitrary } a \end{aligned} \quad (2.2)$$

2.1.2 Convolution Sum of DT Signal

1. Definition

Convolution Sum of two discrete-time sequence $x[n]$ and $y[n]$ is defined by

$$z[n] = x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k] y[n-k] \quad (2.3)$$

- Step. 1) $y[k]$ is time-reversed, then shifted by n ; $y[k] \rightarrow y[-k] \rightarrow y[n-k]$
 Step. 2) $x[k]$ and $y[n-k]$ are multiplied, then summed over all k
 Step. 3) Convolution will remain zero as long as x & y do not overlap
 Step. 4) Sweep $y[n-k]$ from $n = -\infty$ to $n = \infty$ to produce the entire output

2. Properties of the Convolution Sum

The convolution sum has the following properties. Refer [Schaum's text, Problem 2.26] for the proof.

a) Commutative

$$x[n] * y[n] = y[n] * x[n]$$

b) Associative

$$\{x[n] * y_1[n]\} * y_2[n] = x[n] * \{y_1[n] * y_2[n]\}$$

c) Distributive

$$x[n] * \{y_1[n] + y_2[n]\} = x[n] * y_1[n] + x[n] * y_2[n]$$

3. Additional Properties

Refer [Schaum's text, Problem 2.27, 2.31] for the proof.

a) $x[n] * \delta[n] = x[n]$

b) $x[n] * \delta[n - n_0] = x[n - n_0]$

c) $x[n] * u[n] = \sum_{k=-\infty}^n x[k]$

d) $x[n] * u[n - n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$

e) If $x[n]$ and $y[n]$ are periodic sequence with a common period N , the convolution in (2.3) does not converge. Instead, we define the **periodic convolution** $f[n] = x[n] \circledast y[n]$, where $f[n]$ is periodic with period N .

$$f[n] = x[n] \circledast y[n] = \sum_{k=0}^{N-1} x[k] y[n-k] \quad (2.4)$$

[Example 2-1] Evaluate the following convolutions

1. $u(t+a) * u(t+b)$

2. $\text{rect}(t/\tau) * \text{rect}(t/\tau)$

3. $\text{rect}(t/\tau) * u(t)$

4. $x(t) * y(t)$ where $x(t) = \begin{cases} 1 & \text{for } 0 < t < 3 \\ 0 & \text{otherwise} \end{cases}$ and $y(t) = \begin{cases} 1 & \text{for } 0 < t < 2 \\ 0 & \text{otherwise} \end{cases}$

5. $\text{rect}(t/\tau) * \delta_T(t)$ where $\tau < T$ and $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ is the **unit impulse train**

Refer [Schaum's text, Problem 2.6, 2.7, 2.8]

Solution) Example 2-1. 1) The convolution integral is given by

$$\begin{aligned} u(t+a) * u(t+b) &= \int_{-\infty}^{\infty} u(\tau+a) u(-\tau+b+t) d\tau = (t+a+b) u(t+a+b) \\ &= \begin{cases} \int_{-a}^{b+t} 1 \cdot d\tau = (t+a+b) & \text{if } t+a+b > 0 \\ 0 & \text{if } t+a+b < 0 \end{cases} \end{aligned} \quad (2.5)$$

where the product of two step functions $u(\tau+a)u(-\tau+b+t)$ has a non-zero value at $\tau+a > 0$ and $-\tau+b+t > 0$. As shown in Fig. 2.1, if $-a < b+t$, then the product of two step functions overlap each other within the interval $-a < \tau < b+t$. If $b+t < -a$, there is no overlap and the integral in (2.5) becomes zero.

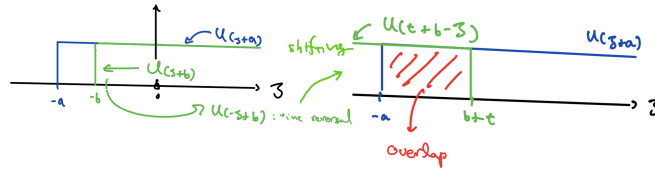


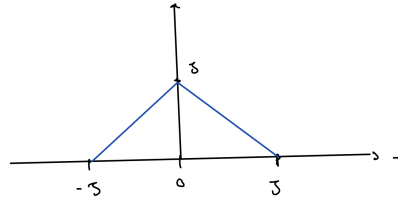
Figure 2.1:

Example 2-1. 2) The convolution can be expanded by expressing the rectangular pulse signal via the step function

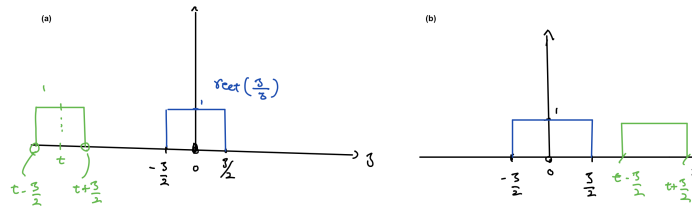
$$\begin{aligned} \text{rect}(t/\tau) * \text{rect}(t/\tau) &= \left\{ u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right) \right\} * \left\{ u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right) \right\} \\ &= (t+\tau)u(t+\tau) - 2tu(t) + (t-\tau)u(t-\tau), \end{aligned} \quad (2.6)$$

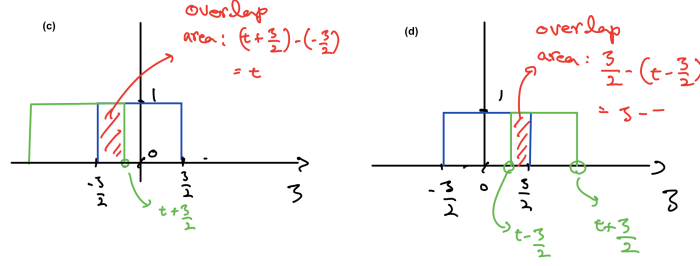
where we used (1.9) in the first equality and the result from [Example 2-1. 1] in the second equality. The last expression has four separate intervals with different values, which is a *triangular pulse signal* with maximum magnitude τ at $t = 0$ and width 2τ (two times larger than the rectangular pulse signal $\text{rect}(t/\tau)$).

$$\begin{cases} 0 & \text{if } t < -\tau \\ t + \tau & \text{if } -\tau < t < 0 \\ t + \tau - 2t = \tau - t & \text{if } 0 < t < \tau \\ \tau - t + t - \tau = 0 & \text{if } t > \tau \end{cases}$$



(b) can also be solved using the direct definition of the convolution where we multiply one original signal to another signal that is time reversed, then time-shifted by t , which is plotted below.





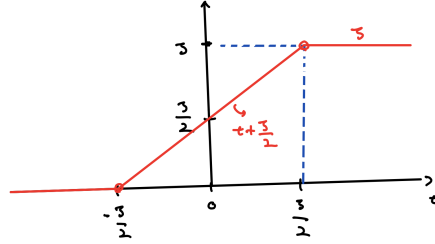
- For sub-figure (a) and (b), there is no overlap between two signals, hence the convolution is zero. These cases correspond to the condition $t + \frac{\tau}{2} < -\frac{\tau}{2}$ and $t - \frac{\tau}{2} > \frac{\tau}{2}$, i.e., $t < -\tau$ for sub-figure (a) and $t > \tau$ for (b).
- For sub-figure (c), if $t + \frac{\tau}{2} > -\frac{\tau}{2}$ and $t - \frac{\tau}{2} < -\frac{\tau}{2}$, the overlap area is $t + \frac{\tau}{2} - (-\frac{\tau}{2}) = t$, which is the convolution in the interval $-\tau < t < 0$.
- For sub-figure (d), if $t - \frac{\tau}{2} < \frac{\tau}{2}$ and $t + \frac{\tau}{2} > \frac{\tau}{2}$, the overlap area is $\frac{\tau}{2} - (t - \frac{\tau}{2}) = \tau - t$, which is the convolution in the interval $0 < t < \tau$.

Example 2-1. 3) The convolution can be expanded in terms of the step function as follows

$$\text{rect}(t/\tau) * u(t) = \left\{ u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right) \right\} * u(t) = \left(t + \frac{\tau}{2}\right) u\left(t + \frac{\tau}{2}\right) - \left(t - \frac{\tau}{2}\right) u\left(t - \frac{\tau}{2}\right), \quad (2.7)$$

where we applied [Example 2-1. 1] in the second equality. The last expression has three separate intervals with different values as follows

$$\begin{cases} 0 & \text{if } t < -\frac{\tau}{2} \\ t + \frac{\tau}{2} & \text{if } -\frac{\tau}{2} < t < \frac{\tau}{2} \\ t + \frac{\tau}{2} - \left(t - \frac{\tau}{2}\right) = \tau & \text{if } t > \frac{\tau}{2} \end{cases}$$

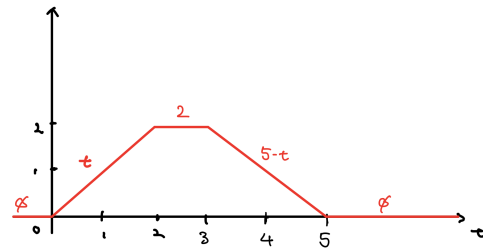


Example 2-1. 4) The convolution can be expanded in terms of the step function as follows

$$\begin{aligned} x(t) * y(t) &= \{u(t) - u(t-3)\} * \{u(t) - u(t-2)\} \\ &= tu(t) - (t-3)u(t-3) - (t-2)u(t-2) + (t-5)u(t-5), \end{aligned} \quad (2.8)$$

where we applied [Example 2-1. 1] in the second equality. The last expression has five separate intervals with different values as follows

$$\begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 < t < 2 \\ t - (t-2) = 2 & \text{if } 2 < t < 3 \\ 2 - (t-3) = 5 - t & \text{if } 3 < t < 5 \\ 5 - t - (t-5) = 0 & \text{if } t > 5 \end{cases}$$



Example 2-1. 5)

$$\text{rect}(t/\tau) * \left[\sum_{n=-\infty}^{\infty} \delta(t - nT) \right] = \sum_{n=-\infty}^{\infty} \text{rect}(t/\tau) * \delta(t - nT) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t - nT}{\tau}\right) \quad (2.9)$$

where we used distributive property in the second equality and $x(t) * \delta(t - t_0) = x(t - t_0)$ in the last equality.

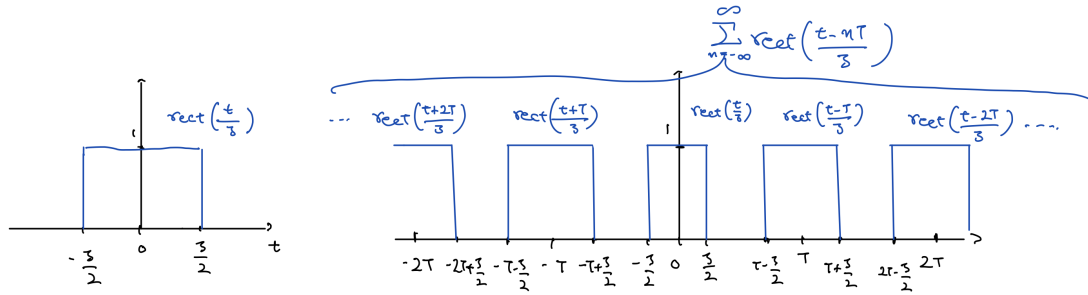


Figure 2.2:

As shown in Fig. 2.2, the time-shifted rectangular pulse signal do not overlap each other as far as $\tau < T$ is satisfied. However, if $\tau > T$, then the time-shifted rectangular pulse signals will overlap each other.

Important Convolution Pairs

1. $u(t+a) * u(t+b) = (t+a+b) u(t+a+b)$
2. $\text{rect}(t/\tau) * \text{rect}(t/\tau) = \begin{cases} 0 & \text{if } t < -\tau \\ \tau+t & \text{if } -\tau < t < 0 \\ \tau-t & \text{if } 0 < t < \tau \\ 0 & \text{if } t > \tau \end{cases}$

[Example 2-2] Evaluate the following convolutions

1. $x(t) * y(t)$ where $x(t) = u(t)$ and $y(t) = e^{-\alpha t} u(t)$, $\alpha > 0$
2. $x(t) * y(t)$ where $x(t) = e^{-\alpha t} u(t)$ and $y(t) = e^{\alpha t} u(-t)$, $\alpha > 0$

Refer [Schaum's text, Problem 2.4, 2.5]

Solution) Example 2-2. 1)

$$\begin{aligned} x(t) * y(t) &= \int_{-\infty}^{\infty} e^{-\alpha \tau} u(\tau) u(t-\tau) d\tau = \begin{cases} \int_0^t e^{-\alpha \tau} d\tau = \frac{1}{\alpha} (1 - e^{-\alpha t}), & \text{if } t > 0 \\ 0, & \text{if } t < 0 \end{cases} \\ &= \frac{1}{\alpha} (1 - e^{-\alpha t}) u(t) \end{aligned} \quad (2.10)$$

Example 2-2. 2)

$$x(t) * y(t) = \int_{-\infty}^{\infty} e^{\alpha \tau} u(-\tau) e^{-\alpha(t-\tau)} u(t-\tau) d\tau \quad (2.11)$$

where the product of two step functions $u(-\tau) u(t-\tau)$ is determined by the magnitude of t as shown in Fig. 2.3. Then (2.11) can be derived as follows

$$\begin{cases} e^{-\alpha t} \int_{-\infty}^0 e^{2\alpha \tau} d\tau = \frac{1}{2\alpha} e^{-\alpha t}, & \text{if } t > 0 \\ e^{-\alpha t} \int_{-\infty}^t e^{2\alpha \tau} d\tau = \frac{1}{2\alpha} e^{\alpha t}, & \text{if } t < 0 \end{cases} \Rightarrow \frac{1}{2\alpha} e^{-\alpha|t|} \quad (2.12)$$

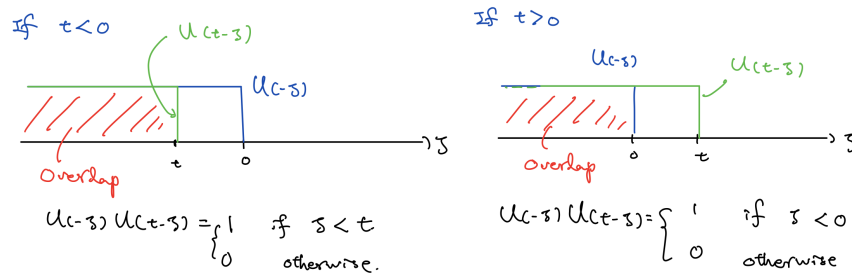


Figure 2.3:

2.2 LTI System Response

Linear Time-Invariant (LTI) System, represented by $T\{\cdot\}$, satisfy the following two attributes.

- Linearity: $T\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \alpha_1 T\{x_1(t)\} + \alpha_2 T\{x_2(t)\}$
- Time-Invariance: $T\{x(t - t_0)\} = y(t - t_0)$

2.2.1 Response of a CT LTI System

1. Impulse Response

- **Impulse Response** is defined as the output of a system when the input is a impulse signal $\delta(t)$.

$$h(t) = T\{\delta(t)\} \quad (2.13)$$

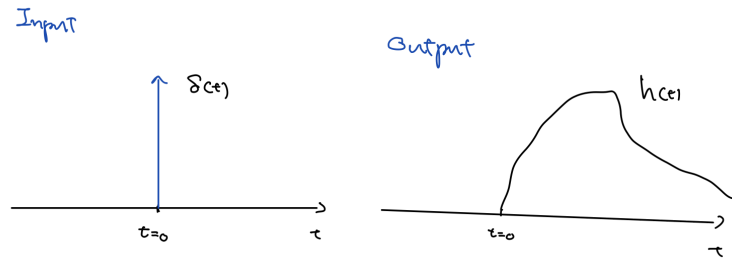


Figure 2.4:

- The output of any CT-LTI system is the convolution of the input $x(t)$ with the impulse response $h(t)$

$$y(t) = x(t) * h(t) \quad (2.14)$$

Proof) Arbitrary input $x(t)$ can be expressed in terms of the impulse signal $\delta(t)$ using convolution

$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau \quad (2.15)$$

Using the properties of an LTI system, the output to an arbitrary input $x(t)$ can be expressed as

$$\begin{aligned} y(t) &= T\{x(t)\} = T\left\{\int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau\right\} \\ &= \int_{-\infty}^{\infty} x(\tau) T\{\delta(t - \tau)\} d\tau = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = x(t) * h(t), \end{aligned} \quad (2.16)$$

by using (2.15) in the second equality, linearity in the third, and time-invariance in the fourth equality.

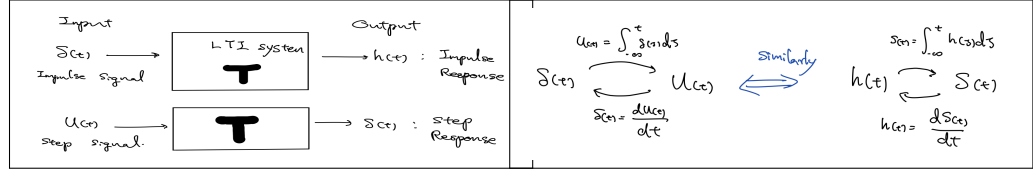
2. Step Response

- **Step Response** is defined as the output of a system when the input is a step signal $u(t)$.

$$s(t) = T\{u(t)\} \quad (2.17)$$

- **Step response** can be obtained by integrating the impulse response $h(t)$. Similarly, the impulse response $h(t)$ can be determined by differentiating the step response $s(t)$.

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau) u(t - \tau) d\tau = \int_{-\infty}^t h(\tau) d\tau \Leftrightarrow h(t) = \frac{ds(t)}{dt} \quad (2.18)$$



3. Properties of CT LTI System

- **Memoryless or Memory System:** The output of a memoryless system depends only on the present input, so that the input-output relationship can be represented in the form $y(t) = Kx(t)$. Since $h(t)$ is the system output for an impulse signal input $\delta(t)$, the impulse response $h(t)$ can be expressed as $h(t) = K\delta(t)$. Hence, the following statement holds

$$\text{If } h(t_0) \neq 0 \text{ for } t_0 \neq 0, \text{ then it is a LTI system with memory.} \quad (2.19)$$

- **Causality:** For a causal system, the output at a present instant do not anticipate input from future instants. In other words, the input that occurs before $t < t_0$ can only determine the output with $t < t_0$.

$$x(t), t < t_0 \Leftrightarrow y(t), t < t_0 \quad (2.20)$$

Thus, in a causal system, it is impossible to obtain an output before an input is applied. Since the impulse response is defined as $h(t) = T\{\delta(t)\}$, the impulse response $h(t)$ is zero for $t < 0$

$$\delta(t) = 0, t < 0 \Leftrightarrow h(t) = 0, t < 0 \quad (2.21)$$

Due to (2.21), the output of a causal LTI system can be expressed as follows

$$\text{Causal LTI System, Arbitrary Input: } y(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau = \int_{-\infty}^t x(\tau) h(t - \tau) d\tau \quad (2.22)$$

Furthermore, if we define a *causal signal* to achieve the following condition

$$x(t) = 0, t < 0, \quad (2.23)$$

the output of causal signal input on a causal LTI system can be expressed as follows

$$\text{Causal LTI System, Causal Input: } y(t) = \int_{-\infty}^t h(\tau) x(t - \tau) d\tau = \int_0^t x(\tau) h(t - \tau) d\tau \quad (2.24)$$

- **Stability:** An LTI system is stable if its impulse response $h(t)$ is absolutely integrable.

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \quad (2.25)$$

Proof) If $|x(t)| \leq k_1 < \infty$ for any t , then the output can be bounded as follows

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \right| \leq \int_{-\infty}^{\infty} |h(\tau) x(t-\tau)| d\tau \\ &= \int_{-\infty}^{\infty} |h(\tau)| |x(t-\tau)| d\tau \leq k_1 \int_{-\infty}^{\infty} |h(\tau)| d\tau \end{aligned} \quad (2.26)$$

If $\int_{-\infty}^{\infty} |h(\tau)| d\tau \leq k_2 < \infty$, then $|y(t)| \leq k_1 \times k_2 < \infty$ and the system is stable.

2.2.2 Response of a DT LTI System

1. Impulse Response

- *Impulse Response* is the output of a system when the input is a impulse signal $\delta[n]$

$$h[n] = \mathbf{T}\{\delta[n]\} \quad (2.27)$$

- *The output of any DT-LTI system is the convolution of the input $x[n]$ with the impulse response $h[n]$*

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \quad (2.28)$$

2. Step Response

- *Step Response* is the system output for a step signal input $u[n]$, i.e., $s[n] = \mathbf{T}\{u[n]\}$
- *Step response* can be obtained by calculating the cumulative sum of the impulse response $h[n]$. Similarly, the impulse response $h[n]$ can be determined by differentiating the step response $s[n]$.

$$\begin{aligned} s[n] &= h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k] u[n-k] = \sum_{k=-\infty}^n h[k], \\ h[n] &= s[n] - s[n-1] \end{aligned} \quad (2.29)$$

3. Properties of DT LTI System

- **Memory System**

$$\text{If } h[n_0] \neq 0 \text{ for } n_0 \neq 0, \text{ then it is a LTI system with memory.} \quad (2.30)$$

- **Causality:** The causality condition for a DT-LTI system is given by

$$h[n] = 0, \quad n < 0 \quad (2.31)$$

Due to (2.31) and the definition of causal signal, the output of a causal LTI system can be expressed as

$$\text{Causal LTI System, Arbitrary Input: } y[n] = \sum_{k=0}^{\infty} h[k] x[n-k] = \sum_{k=-\infty}^n x[k] h[n-k], \quad (2.32)$$

$$\text{Causal LTI System, Causal Input: } y[n] = \sum_{k=0}^n h[k] x[n-k] = \sum_{k=0}^n x[k] h[n-k]$$

- **Stability:** An LTI system is stable if its impulse response $h(t)$ is absolutely summable.

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \quad (2.33)$$

[Example 2-3] Consider a CT-LTI system with step response $s(t) = e^{-t}u(t)$. Determine the output of the system for input signal $x(t) = u(t-1) - u(t-3)$.

Refer [Schaum's text, Problem 2.10]

Solution Based on the definition of a step response $s(t) = \mathcal{T}\{u(t)\}$, the output of the LTI system is given by

$$y(t) = s(t-1) - s(t-3) = e^{-(t-1)}u(t-1) - e^{-(t-3)}u(t-3), \quad (2.34)$$

where we used linearity and time-invariance of the LTI system in the second equality.



[Example 2-4] Consider a CT-LTI system described by

$$y(t) = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} x(\tau) d\tau. \quad (2.35)$$

Find the impulse response $h(t)$ of the system and answer whether this system is causal or not.

Refer [Schaum's text, Problem 2.11]

Solution Since the impulse response is the output for the impulse signal input, the following equality holds

$$h(t) = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \delta(\tau) d\tau = \begin{cases} \frac{1}{T} & \text{if } t - \frac{T}{2} < 0 \text{ and } t + \frac{T}{2} > 0, \\ 0 & \text{otherwise} \end{cases} \quad (2.36)$$

where we applied the sifting property of $\delta(t)$ in the second equality. The impulse response $h(t)$ is plotted in Fig. 2.5. Since $h(t) \neq 0$ for $-\frac{T}{2} < t < 0$, this system is not causal.

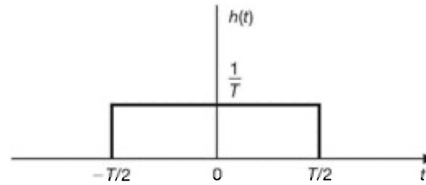


Figure 2.5:



[Example 2-5] Consider CT-LTI systems composed of two component blocks where the impulse response of each block is given by $h_1(t) = e^{-2t}u(t)$ and $h_2(t) = 2e^{-t}u(t)$. For (a) cascade and (b) parallel connection case, find the impulse response $h(t)$ of the overall system and answer whether the overall system is stable or not.

Refer [Schaum's text, Problem 2.14, 2.53]

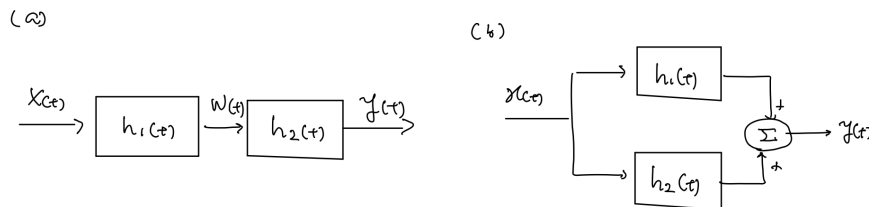


Figure 2.6:

Solution) For (a) cascaded connection case, the impulse response can be derived as follows

$$\begin{aligned} h(t) &= h_1(t) * h_2(t) = \int_{-\infty}^{\infty} 2e^{-\tau} u(\tau) e^{-2(t-\tau)} u(t-\tau) d\tau \\ &= 2e^{-2t} \int_{-\infty}^{\infty} e^{\tau} u(\tau) u(t-\tau) d\tau = \begin{cases} 2e^{-2t} \int_0^t e^{\tau} d\tau = 2(e^{-t} - e^{-2t}) & \text{if } t > 0, \\ 0 & \text{otherwise} \end{cases} \\ &= 2(e^{-t} - e^{-2t}) u(t) \end{aligned} \quad (2.37)$$

To check stability, we need to verify whether $h(t)$ is absolutely integrable.

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_0^{\infty} 2(e^{-\tau} - e^{-2\tau}) d\tau = 2 \left[\int_0^{\infty} e^{-\tau} d\tau - \int_0^{\infty} e^{-2\tau} d\tau \right] = 1 < \infty \quad (2.38)$$

For (b) parallel connection case, the overall impulse response is given by

$$h(t) = h_1(t) + h_2(t) = (e^{-2t} + 2e^{-t}) u(t), \quad (2.39)$$

and the stability test can be performed as below

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_0^{\infty} (e^{-2\tau} + 2e^{-\tau}) d\tau = \frac{1}{2} + 2 < \infty, \quad (2.40)$$

which indicates that the parallel connection system in (b) is stable.



[Example 2-6] For the following impulse responses, determine whether the given LTI system is causal and stable.

- | | |
|----------------------------------|--|
| a) $h(t) = e^{-3t} \sin(t) u(t)$ | b) $h(t) = \delta(t) + e^{-3t} u(t)$ |
| c) $h[n] = \delta[n+1]$ | d) $h[n] = \left(-\frac{1}{2}\right)^n u[n-1]$ |

Solution) For causality, we need to check whether the impulse response has a non-zero value on $t < 0$ (or $n < 0$). Then, it is clear that, except (c), the other systems are all causal system. For stability test, we need to check whether the given impulse response is absolutely integrable (or absolutely summable).

$$\begin{aligned} \text{(a)} \quad & \int_0^{\infty} e^{-3t} |\sin(t)| dt \leq \int_0^{\infty} e^{-3t} dt = \frac{1}{3} < \infty \\ \text{(b)} \quad & \int_{-\infty}^{\infty} |h(\tau)| d\tau = 1 + \int_0^{\infty} e^{-3\tau} d\tau = \frac{4}{3} < \infty \\ \text{(c)} \quad & \sum_{n=-\infty}^{\infty} |h[n]| = 1 < \infty, \quad \text{(d)} \quad \sum_{n=1}^{\infty} \left| \left(-\frac{1}{2}\right)^n \right| = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 < \infty \end{aligned} \quad (2.41)$$

Hence, the solutions are summarized below.

- | | |
|----------------------|-------------------|
| a) Causal, Stable | b) Causal, Stable |
| c) Noncausal, Stable | d) Causal, Stable |



[Example 2-7] Compute the output $y[n]$ of a DT-LTI system for the given impulse response and the input signals

- | | |
|---|--|
| a) $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n]$ | b) $x[n] = u[n]$ and $h[n] = \alpha^n u[n]$, where $0 < \alpha < 1$ |
|---|--|

Refer [Schaum's text, Problem 2.28, 2.29]

Solution) In (a), the convolution sum between the input signal and the impulse response is given by

$$\begin{aligned}
 y[n] &= \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} \alpha^k \beta^{n-k} u[k] u[n-k] \\
 &= \beta^n \sum_{k=0}^n \left(\frac{\alpha}{\beta} \right)^k u[n] = \begin{cases} \beta^n \frac{1 - (\alpha/\beta)^{n+1}}{1 - \alpha/\beta} u[n] = \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} u[n] & \text{if } \alpha \neq \beta \\ \beta^n (n+1) u[n] & \text{if } \alpha = \beta \end{cases} \quad (2.42)
 \end{aligned}$$

For (b), we can apply [Example 2-7. a] by substituting $\alpha \leftarrow 1$ and $\beta \leftarrow \alpha$ into (2.42). Then, the output is given by

$$y[n] = \frac{1 - \alpha^{n+1}}{1 - \alpha} u[n] \quad (2.43)$$



[Example 2-8] Compute the DT convolution of the following sequences

- $x[0] = 0.5, x[1] = 2, x[n] = 0$ otherwise.
- $h[0] = h[1] = h[2] = 1, h[n] = 0$ otherwise.

Compute $y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$.

Solution) Using the convolution sum $y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$

$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1] \quad (2.44)$$

Substituting the values of $h[n]$ into $y[n]$, we have

- $y[0] = 0.5h[0] + 2h[0-1] = 0.5 \times 1 = 0.5$
- $y[1] = 0.5h[1] + 2h[0] = 0.5 \times 1 + 2 \times 1 = 2.5$
- $y[2] = 0.5h[2] + 2h[1] = 0.5 \times 1 + 2 \times 1 = 2.5$
- $y[3] = 0.5h[3] + 2h[2] = 0.5 \times 0 + 2 \times 1 = 2$
- $y[n] = 0$ for $n < 0$ and $n \geq 4$.



3 Fourier Series

Major References:

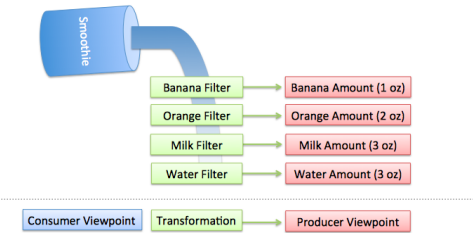
- Chapter 3, *Signals and Systems* by Alan V. Oppenheim et. al., 2nd edition, Prentice Hall
- Chapter 5.2 & 6.2, *Schaum's Outline of Signals and Systems*, 2nd Edition, 2010, McGraw-Hill

3.1 Introduction



Jean-Baptiste Joseph Fourier (1768-1830)

Smoothie to Recipe



1st Metaphor of the Fourier Analysis (Source: <https://betterexplained.com>)

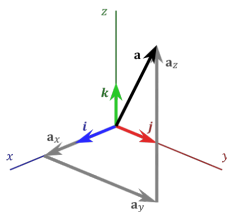
What is Fourier Analysis?

1. **What does the Fourier Transform do?**
Given a smoothie, it finds the recipe.
2. **How?**
Run the smoothie through filters to extract each ingredient.
3. **Why?** Recipes are easier to analyze, compare, and modify than the smoothie itself.
4. **How do we get the smoothie back?**
Blend the ingredients.

Important Points to consider

1. **Filters must be independent**
2. **Filters must be complete**
3. **Ingredients must be combineable.**
The ingredients must make the same result when separated and combined in any order.

2nd Metaphor of the Fourier Analysis



$$\mathbf{a} = a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z$$

- Arbitrary vector can be expressed via unit vectors and the magnitude toward each unit vector.
- Can we break a function into its simple functions (referred to as **base functions**) just like vector case?
- Can we combine the **base functions** to represent arbitrary signals?

$$\Rightarrow x(t) = \sum_{n=-\infty}^{\infty} a_n \psi_n(t), \quad \text{where } \psi_n(t) \text{ is the base function.}$$

1. **Fourier analysis** is the study of the way general functions may be represented or approximated by sums of simpler trigonometric functions.
 - **Analysis**: breaking up a signal into simpler constituent parts
 - **Synthesis**: reassembling a signal from its constituent parts
 ⇒ **Fourier analysis is all about breaking & reassembling a function**
2. **Fourier Series**: fourier analysis for periodic signals
3. **Fourier Transform**: fourier analysis for non-periodic signals

	Periodic signal	Aperiodic Signal
Continuous Time	Fourier Series (FS)	Fourier Transform (FT)
Discrete Time	Discrete time FS	Discrete time FT

3.2 Continuous Time Fourier Series

For a periodic signal $x(t)$ with fundamental period T_0 , we adopt sinusoidal signals as the base function

$$\left\{ \begin{array}{l} \text{fundamental period: } T_0, \quad \text{fundamental frequency: } f_0 = \frac{1}{T_0}, \\ \text{fundamental angular frequency: } \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} \end{array} \right\}$$

Then the CT-Fourier series can be expressed into the following two representations. All of the proof for Chapter 3.2 are summarized at the end of the section.

1. Fourier Series (Complex Exponential Series Form)

The base function for this form is $\psi_k(t) = e^{jk\omega_0 t} = e^{j2\pi k f_0 t}$

1. **Synthesis**:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \psi_k(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad (3.1)$$

2. **Analysis**

$$c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_{T_0} x(t) \psi_k^*(t) dt, \quad (3.2)$$

where the integration interval T_0 is any period with length T_0 , e.g., $[0, T_0]$ or $[-\frac{T_0}{2}, \frac{T_0}{2}]$

[Properties]

1. The set of base functions $\{\psi_k(t)\}$ is orthogonal on any interval over a period T_0 , $(\alpha, \alpha + T_0)$

$$\int_{\alpha}^{\alpha+T_0} \psi_m(t) \psi_k^*(t) dt = \begin{cases} 0, & m \neq k \\ T_0, & m = k \end{cases} \quad (3.3)$$

2. If $x(t)$ is a real function, then $c_{-k} = c_k^*$

By using the Euler's Formula, $e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$, the Fourier series in the complex exponential series form can be converted to a trigonometric series form as follows.

2. Fourier Series (Trigonometric Series Form)

1. Synthesis

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)) \quad (3.4)$$

2. Analysis

$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos(k\omega_0 t) dt, \quad b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin(k\omega_0 t) dt \quad (3.5)$$

[Properties]

1. The conversion between two representations

$$\begin{cases} \frac{a_0}{2} = c_0, \\ a_k = c_k + c_{-k}, \quad b_k = j(c_k - c_{-k}) \end{cases} \Leftrightarrow \begin{cases} c_k = \frac{1}{2}(a_k - jb_k), \\ c_{-k} = \frac{1}{2}(a_k + jb_k), \end{cases} \quad (3.6)$$

2. If $x(t)$ is a real function, then $a_k = 2 \operatorname{Re}[c_k]$, $b_k = -2 \operatorname{Im}[c_k]$.

A periodic signal $x(t)$ has a Fourier series representation if it satisfies the Dirichlet conditions. In other words, Dirichlet conditions are the sufficient conditions (but not necessary condition) for the Fourier series to converge.



Peter Gustav Lejeune Dirichlet (1805-1859)



3. Dirichlet Condition (Sufficient conditions for FS to exist)

1. $x(t)$ is absolutely integrable over any period $\int_{T_0} |x(t)| dt < \infty$
2. $x(t)$ has a finite number of maxima and minima within any finite interval of t .
3. $x(t)$ has a finite number of discontinuities within any finite interval of t , and each of these discontinuities is finite.

If $x(t)$ satisfies the Dirichlet condition, then the corresponding Fourier series is convergent and its sum is $x(t)$, except at any point t_0 at which $x(t)$ is discontinuous.

$$x(t_0) = \frac{1}{2} [x(t_0^+) + x(t_0^-)]$$

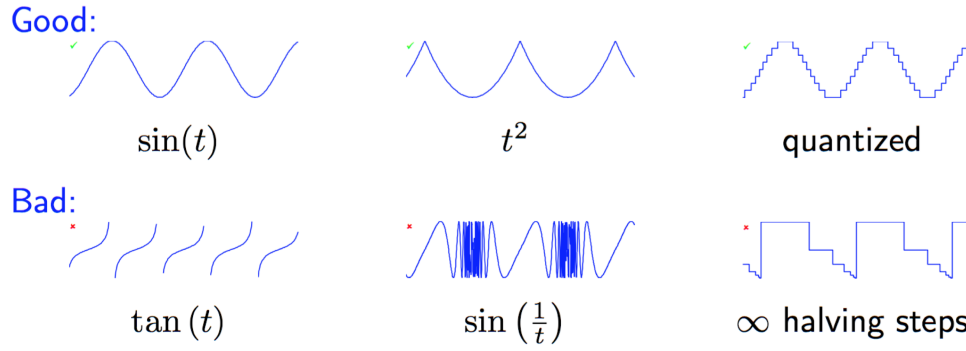


Figure 3.1: Examples of the Dirichlet Condition

4. Parseval's Theorem

Average signal power can be calculated by integral over time domain or infinite sum over frequency domain.

$$P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad (3.7)$$

[Proof of Chapter 3.2]

1. Prove that the set $\{\psi_k(t)\} = \{e^{jk\omega_0 t}\}$ of base functions is orthogonal.

Proof) Refer [Schaum's text, Problem 5.1]

$$\int_{\alpha}^{\alpha+T_0} e^{jm\omega_0 t} e^{-jk\omega_0 t} dt = \int_{\alpha}^{\alpha+T_0} e^{j(m-k)\omega_0 t} dt$$

If $m = k$, then the complex exponential in the integral will be one ($e^0 = 1$) and the integration results will be T_0 . If $m \neq k$, then denote $m - k = l$, which is a nonzero integer, and the integral can be further expressed as

$$\int_{\alpha}^{\alpha+T_0} e^{jl\omega_0 t} dt = \frac{1}{jl\omega_0} e^{jl\omega_0 t} \Big|_{\alpha}^{\alpha+T_0} = \frac{e^{j\alpha l\omega_0 T_0}}{jl\omega_0} [e^{j2\pi l} - 1] = 0, \quad (3.8)$$

where the second equality follows by $\omega_0 = 2\pi/T_0$.

2. Derive $c_k = \frac{1}{T_0} \int_{T_0} x(t) \psi_k^*(t) dt$ using the orthogonality condition of $\{\psi_k(t)\}$.

Proof) Refer [Schaum's text, Problem 5.2]

$$\begin{aligned} \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) \psi_k^*(t) dt &= \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} \left[\sum_{m=-\infty}^{\infty} c_m \psi_m(t) \right] \psi_k^*(t) dt \\ &= \frac{1}{T_0} \sum_{m=-\infty}^{\infty} c_m \underbrace{\int_{\alpha}^{\alpha+T_0} \psi_m(t) \psi_k^*(t) dt}_{=T_0 \delta_{m,k}} = c_k, \end{aligned} \quad (3.9)$$

where $\delta_{m,k} = 1$ if $m = k$ and $\delta_{m,k} = 0$ if $m \neq k$.

3. Derive the trigonometric series representation of FS from the complex exponential form.

Proof) Apply the Euler Formula $e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$ into the complex exponential FS formula as follows. Refer [Schaum's text, Problem 5.3].

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = c_0 + \sum_{k=1}^{\infty} [c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t}] \\ &= c_0 + \sum_{k=1}^{\infty} [(c_k + c_{-k}) \cos(k\omega_0 t) + j(c_k - c_{-k}) \sin(k\omega_0 t)] \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)), \end{aligned} \quad (3.10)$$

where $\frac{a_0}{2} = c_0$, $a_k = c_k + c_{-k}$, and $b_k = j(c_k - c_{-k})$.

4. Derive the Parseval's theorem in (3.7).

Proof) Let's evaluate the power of a periodic signal $x(t)$. Refer [Schaum's text, Problem 5.14].

$$\begin{aligned} P &= \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \frac{1}{T_0} \int_{T_0} x(t) x^*(t) dt \\ &= \frac{1}{T_0} \int_{T_0} \left\{ \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \times \sum_{m=-\infty}^{\infty} c_m^* e^{-jm\omega_0 t} \right\} dt \quad \left(\text{Applied } x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right) \\ &= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_k c_m^* \underbrace{\int_{T_0} e^{jk\omega_0 t} e^{-jm\omega_0 t} dt}_{=T_0 \delta_{k,m}} = \sum_{k=-\infty}^{\infty} |c_k|^2 \end{aligned} \quad (3.11)$$

[Example 3-1] Derive the complex exponential FS representation of the following signals

Refer [Schaum's text, Problem 5.4]

- | | |
|---------------------------------|------------------------------|
| a) $x(t) = \cos(\omega_0 t)$ | b) $x(t) = \sin(\omega_0 t)$ |
| c) $x(t) = \cos(4t) + \sin(6t)$ | d) $x(t) = \sin^2(t)$ |

Solution) For sinusoidal functions, we can get the FS by simply expanding it in terms of the complex exponentials.

$$\begin{aligned} \cos(\omega_0 t) &= \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t}), \quad \sin(\omega_0 t) = \frac{1}{2j} (e^{j\omega_0 t} - e^{-j\omega_0 t}), \\ \cos(4t) + \sin(6t) &= \frac{1}{2} (e^{j4t} + e^{-j4t}) + \frac{1}{2j} (e^{j6t} - e^{-j6t}), \\ \sin^2(t) &= \frac{1}{2} (1 - \cos 2t) = \frac{1}{2} - \frac{1}{4} e^{j2t} - \frac{1}{4} e^{-j2t} \end{aligned} \quad (3.12)$$

Hence, the complex Fourier coefficients for each signals are derived as follows.

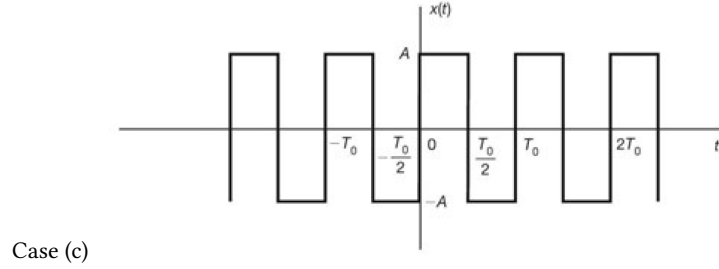
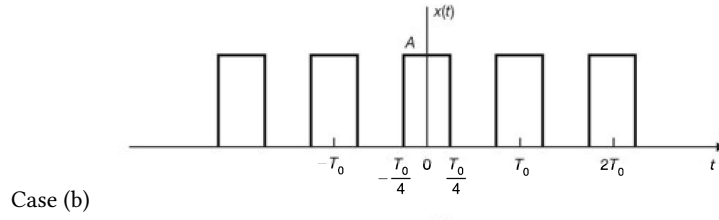
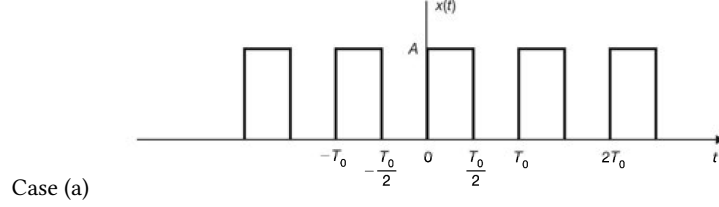
- (a) $c_1 = \frac{1}{2}$, $c_{-1} = \frac{1}{2}$, and $c_k = 0$ for other k index.
 (b) $c_1 = \frac{1}{2j}$, $c_{-1} = -\frac{1}{2j}$, and $c_k = 0$ for other k index.
 (c) $c_{-3} = -\frac{1}{2j}$, $c_{-2} = \frac{1}{2}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{1}{2j}$, and $c_k = 0$ for other k index.
 (d) $c_1 = -\frac{1}{4}$, $c_{-1} = -\frac{1}{4}$, $c_0 = \frac{1}{2}$, and $c_k = 0$ for other k index.



[Example 3-2] Determine the complex and trigonometric FS representation of the following signals.
Refer [Schaum's text, Problem 5.5, 5.6, 5.7]

- a) Case (a)
c) Case (c)

b) Case (b)



Solution) For (a), the Complex Fourier coefficients can be derived as follows

$$\begin{aligned}
 c_0 &= \frac{A}{T_0} \int_0^{T_0/2} dt = \frac{A}{2} \\
 c_k &= \frac{A}{T_0} \int_0^{T_0/2} e^{-jk\omega_0 t} dt = \frac{A}{jk\omega_0 T_0} e^{-jk\omega_0 t} \Big|_0^{T_0/2} = \frac{A}{jk\omega_0 T_0} [1 - e^{-jk\pi}] = \frac{A}{j2\pi k} [1 - (-1)^k] \\
 &= \begin{cases} 0 & \text{for even } k = 2m, \\ \frac{A}{j\pi(2m+1)} & \text{for odd } k = 2m+1 \end{cases}
 \end{aligned} \tag{3.13}$$

The trigonometric Fourier coefficients can be derived as follows.

$$\begin{aligned}
 a_0 &= 2c_0 = A, \quad a_k = 2 \operatorname{Re} [c_k] = 0, \\
 b_k &= -2 \operatorname{Im} [c_k] = \begin{cases} 0 & \text{for even } k = 2m, \\ \frac{2A}{(2m+1)\pi} & \text{for odd } k = 2m+1 \end{cases}
 \end{aligned} \tag{3.14}$$

The signals $x_b(t)$ for case (b) is a time-shifted version of the case (a), i.e., $x_b(t) = x_a\left(t + \frac{T_0}{4}\right)$. Then, the Fourier coefficient of the time-shifted signal is derived as follows

$$\frac{1}{T_0} \int_{T_0} x\left(t + \frac{T_0}{4}\right) e^{-jk\omega_0 t} dt = \frac{1}{T_0} e^{jk\omega_0 \frac{T_0}{4}} \int_{T_0} x(\tau) e^{-jk\omega_0 \tau} d\tau = c_k e^{jk\frac{\pi}{2}} = j^k c_k, \tag{3.15}$$

where we used a change of variable, i.e., $\tau = t + \frac{T_0}{4}$. Hence, the Fourier coefficients of case (b) are given by

$$\begin{aligned}
 c_0 &= \frac{A}{2} \times j^0 = \frac{A}{2} \\
 c_k &= \begin{cases} 0 & \text{for even } k = 2m, \\ \frac{A \times j^{2m+1}}{j\pi(2m+1)} = \frac{A(-1)^m}{\pi(2m+1)} & \text{for odd } k = 2m+1 \end{cases} \\
 a_0 &= 2c_0 = A, \quad b_k = -2\operatorname{Im}[c_k] = 0, \\
 a_k &= 2\operatorname{Re}[c_k] = \begin{cases} 0 & \text{for even } k = 2m, \\ \frac{2A(-1)^m}{\pi(2m+1)} & \text{for odd } k = 2m+1 \end{cases}
 \end{aligned} \tag{3.16}$$

The signals $x_c(t)$ for case (c) is a scaled version of the case (a), i.e., $x_c(t) = 2x_a(t) - A$. Then, the Fourier coefficient of the case (c) is derived as follows

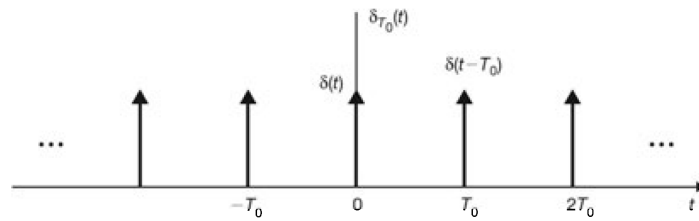
$$\begin{aligned}
 c_0 &= \frac{1}{T_0} \int_{T_0} (2x_a(t) - A) dt = 2[c_0 \text{ for case (a)}] - A \\
 c_k &= \frac{1}{T_0} \int_{T_0} (2x_a(t) - A) e^{-jk\omega_0 t} dt = 2[c_k \text{ for case (a)}],
 \end{aligned} \tag{3.17}$$

where $\int_{T_0} e^{-jk\omega_0 t} dt = 0$. Hence, the Fourier coefficients of case (c) are given by

$$\begin{aligned}
 c_0 &= 2 \times \frac{A}{2} - A = 0, \quad c_k = \begin{cases} 0 & \text{for even } k = 2m, \\ \frac{2A}{j\pi(2m+1)} & \text{for odd } k = 2m+1 \end{cases} \\
 a_0 &= 2c_0 = 0, \quad a_k = 2\operatorname{Re}[c_k] = 0, \\
 b_k &= -2\operatorname{Im}[c_k] = \begin{cases} 0 & \text{for even } k = 2m, \\ \frac{4A}{(2m+1)\pi} & \text{for odd } k = 2m+1 \end{cases}
 \end{aligned} \tag{3.18}$$



[Example 3-3] Determine the complex and trigonometric FS representation of the periodic impulse trains $\delta_{T_0}(t)$ signals, which is defined by $\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$. Refer [Schaum's text, Problem 5.8]



Solution) The Complex Fourier coefficients can be derived as follows

$$c_0 = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t) dt = \frac{1}{T_0}, \quad c_k = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{k=-\infty}^{\infty} \delta(t - kT_0) e^{-jk\omega_0 t} dt = \frac{1}{T_0} \tag{3.19}$$

The trigonometric Fourier coefficients can be derived as follows.

$$a_0 = 2c_0 = \frac{2}{T_0}, \quad a_k = 2 \operatorname{Re} [c_k] = \frac{2}{T_0}, \quad b_k = -2 \operatorname{Im} [c_k] = 0 \quad (3.20)$$

Hence, we get

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} = \frac{1}{T_0} + \frac{2}{T_0} \sum_{k=1}^{\infty} \cos(k\omega_0 t)$$



[Example 3-4] Determine the complex and trigonometric FS representation of the periodic signal $x(t)$ defined by

$$x(t) = t^2, \quad -\pi < t < \pi \quad \text{and} \quad x(t + 2\pi) = x(t).$$

Refer [Schaum's text, Problem 5.62]

Solution For the given signal, $T_0 = 2\pi$ and $\omega_0 = \frac{2\pi}{T_0} = 1$. Then, the Fourier coefficients can be derived as

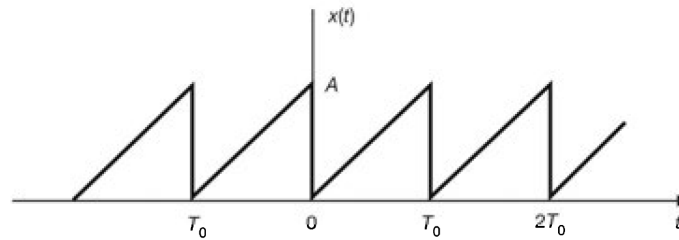
$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 dt = \frac{\pi^2}{3}, \\ c_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 e^{-jkt} dt = \frac{1}{2\pi} \left[\frac{t^2 e^{-jkt}}{jk} \Big|_{-\pi}^{\pi} + \frac{2}{jk} \int_{-\pi}^{\pi} t e^{-jkt} dt \right] \\ &= \frac{\pi}{k} \sin(\pi k t) + \frac{1}{jk\pi} \left[\frac{t e^{-jkt}}{jk} \Big|_{-\pi}^{\pi} + \frac{1}{(jk\pi)^2} e^{-jkt} \Big|_{-\pi}^{\pi} \right] \\ &= \frac{2 \cos(k\pi t)}{k^2} = \frac{2(-1)^k}{k^2}, \\ a_0 &= 2c_0 = \frac{2\pi^2}{3}, \quad b_k = -2 \operatorname{Im} [c_k] = 0, \\ a_k &= 2 \operatorname{Re} [c_k] = \frac{4(-1)^k}{k^2} \end{aligned} \quad (3.21)$$



[Example 3-5] Determine the complex and trigonometric FS representation of the periodic signal $x(t)$ defined by

$$x(t) = \frac{A}{T_0} t, \quad 0 < t < T_0 \quad \text{and} \quad x(t + T_0) = x(t).$$

Refer [Schaum's text, Problem 5.63]



Solution) The Complex Fourier coefficients can be derived as follows

$$\begin{aligned}
 c_0 &= \frac{A}{T_0^2} \int_0^{T_0} t dt = \frac{A}{2}, \\
 c_k &= \frac{A}{T_0^2} \int_0^{T_0} t e^{-jk\omega_0 t} dt = \frac{A}{T_0^2} \left[\frac{t e^{-jk\omega_0 t}}{-jk\omega_0} \Big|_0^{T_0} + \frac{A}{jk\omega_0 T_0^2} \int_0^{T_0} e^{-jk\omega_0 t} dt \right] \\
 &= \frac{Aj}{2\pi k} + \frac{A}{(jk\omega_0 T_0)^2} \left[1 - e^{-j2k\pi} \right] = \frac{Aj}{2\pi k}
 \end{aligned} \tag{3.22}$$

The trigonometric Fourier coefficients can be derived as follows.

$$a_0 = 2c_0 = A, \quad a_k = 2 \operatorname{Re} [c_k] = 0, \quad b_k = -2 \operatorname{Im} [c_k] = -\frac{A}{\pi k} \tag{3.23}$$



✱ **Properties of Fourier Series.** (Refer [Oppenheim], Chapter 3.5 for detailed proof)

1. Linear Property

If the FS coefficients of $x_1(t)$ and $x_2(t)$ are $c_{1,k}$ and $c_{2,k}$, then the FS coefficients of $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ are

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \leftrightarrow \alpha_1 c_{1,k} + \alpha_2 c_{2,k}$$

2. Time Shifting

If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k , the FS coefficients of $x(t - t_0)$ are

$$x(t - t_0) \leftrightarrow e^{-jk\omega_0 t_0} c_k$$

Proof) The FS coefficients of the delayed signal are given by

$$\frac{1}{T_0} \int_{T_0} x(t - t_0) e^{-jk\omega_0 t} dt = \left\{ \frac{1}{T_0} \int_{T_0} x(\tau) e^{-jk\omega_0 \tau} d\tau \right\} \times e^{-jk\omega_0 t_0} = e^{-jk\omega_0 t_0} c_k,$$

where we used a change of variable ($t - t_0 = \tau$) in the first equality and (3.2) in the second equality.

3. Conjugate Property

If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k , the FS coefficients of $x^*(t)$ are given by

$$x^*(t) \leftrightarrow c_{-k}^*$$

If $x(t)$ is a real-valued signal, then $c_k = c_{-k}^*$ follows by the conjugate property.

Proof) By taking the conjugate of (3.2), we get

$$c_k^* = \frac{1}{T_0} \int_{T_0} x(t) e^{jk\omega_0 t} dt, \quad \Rightarrow \quad c_{-k}^* = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt,$$

where the second expression follows by substituting k to $-k$. Then, by comparing (3.2) to the last expression, it follows that $\{c_{-k}^*\}$ are the FS coefficients of the conjugate signal $x^*(t)$.

4. Frequency Shifting

If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k , the FS coefficients of $e^{jk_0 \omega_0 t} x(t)$ are

$$e^{jk_0 \omega_0 t} x(t) \leftrightarrow c_{k-k_0}$$

Proof) Based on (3.2), the Fourier series coefficients of $e^{jk_0 w_0 t} x(t)$ are given by

$$\frac{1}{T_0} \int_{T_0} e^{jk_0 w_0 t} x(t) e^{-jk w_0 t} dt = \frac{1}{T_0} \int_{T_0} x(t) e^{-j(k-k_0) w_0 t} dt = c_{k-k_0}$$

5. Time Reversal Property

If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k , the FS coefficients of $x(-t)$ are given by

$$x(-t) \leftrightarrow c_{-k}$$

6. Time Scaling Property

If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k , the FS coefficients of $x(\alpha t)$, $\alpha > 0$, are

$$x(\alpha t) \leftrightarrow c_k \quad \text{with period } \frac{T_0}{\alpha}$$

Proof) Since the period of $x(\alpha t)$ reduces to $\hat{T}_0 = \frac{T_0}{\alpha}$, the Fourier series coefficients of $x(\alpha t)$ are

$$\frac{1}{\hat{T}_0} \int_{\hat{T}_0} x(\alpha t) e^{-jk \frac{2\pi}{\hat{T}_0} t} dt = \frac{\alpha}{T_0} \int_{\frac{T_0}{\alpha}} x(\alpha t) e^{-jk \frac{2\pi}{T_0} t} dt = \frac{1}{T_0} \int_{T_0} x(\tau) e^{-jk \frac{2\pi}{T_0} \tau} d\tau = c_k,$$

where we used a change of variable ($t = \alpha \tau$) in the second equality.

7. Differentiation

If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k , the FS coefficients of $\frac{dx(t)}{dt}$ are

$$\frac{dx(t)}{dt} \leftrightarrow jk w_0 c_k$$

Proof) By differentiating (3.1), we get

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk w_0 t} \rightarrow \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} jk w_0 c_k e^{jk w_0 t}$$

8. Integration

If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k , the FS coefficients of $\int x(t) dt$ are

$$\int x(t) dt \leftrightarrow \frac{c_k}{jk w_0}$$

Proof) By integrating (3.1), we get

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk w_0 t} \rightarrow \int x(t) dt = \sum_{k=-\infty}^{\infty} \frac{c_k}{jk w_0} e^{jk w_0 t}$$

9. Periodic Convolution

If $x_1(t)$ and $x_2(t)$ are periodic signals with common period T_0 and FS coefficients $c_{1,k}$ and $c_{2,k}$, the FS coefficients of the periodic convolution defined in (2.2) are given by

$$x_1(t) \otimes x_2(t) = \int_{T_0} x_1(\tau) x_2(t - \tau) d\tau \leftrightarrow T_0 c_{1,k} c_{2,k}$$

Proof) The FS coefficients of $x_1(t) \otimes x_2(t)$ are given by

$$\begin{aligned} \frac{1}{T_0} \int_{T_0} x_1(t) \otimes x_2(t) e^{-jk\omega_0 t} dt &= \frac{1}{T_0} \int_{T_0} \left[\int_{T_0} x_1(\tau) x_2(t-\tau) d\tau \right] e^{-jk\omega_0 t} dt \\ &= \frac{1}{T_0} \int_{T_0} \int_{T_0} x_1(\tau) x_2(t_1) d\tau e^{-jk\omega_0 \tau} e^{-jk\omega_0 t_1} dt_1 \\ &= \frac{1}{T_0} \int_{T_0} x_1(\tau) e^{-jk\omega_0 \tau} d\tau \cdot \int_{T_0} x_2(t_1) e^{-jk\omega_0 t_1} dt_1 = T_0 c_{1,k} c_{2,k} \end{aligned}$$

where we used the definition of periodic convolution in the first equality, applied a change of variable ($t-\tau = t_1$) in the second equality, and employed (3.2) for $c_{1,k}$ and $c_{2,k}$ in the last equality

10. Multiplication Property

If $x_1(t)$ and $x_2(t)$ are periodic signals with common period T_0 and FS coefficients $c_{1,k}$ and $c_{2,k}$, the FS coefficients of the product of two signals $x_1(t)x_2(t)$ are given by

$$x_1(t)x_2(t) \leftrightarrow \sum_{l=-\infty}^{\infty} c_{1,l} c_{2,k-l}$$

Proof) The FS coefficients of $x_1(t)x_2(t)$ are given by

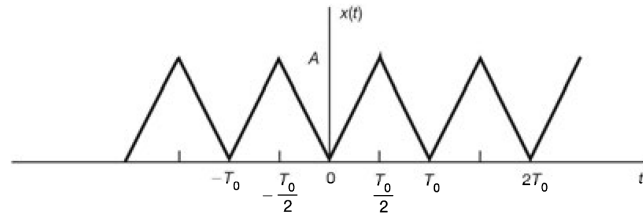
$$\begin{aligned} \frac{1}{T_0} \int_{T_0} x_1(t)x_2(t) e^{-jk\omega_0 t} dt &= \frac{1}{T_0} \int_{T_0} \sum_{l=-\infty}^{\infty} c_{1,l} e^{jl\omega_0 t} x_2(t) e^{-jk\omega_0 t} dt \\ &= \sum_{l=-\infty}^{\infty} c_{1,l} \left[\frac{1}{T_0} \int_{T_0} x_2(t) e^{-j(k-l)\omega_0 t} dt \right] = \sum_{l=-\infty}^{\infty} c_{1,l} c_{2,k-l} \end{aligned}$$

where we applied (3.1) for $x_1(t)$ in the first equality, then used (3.2) in the last equality.

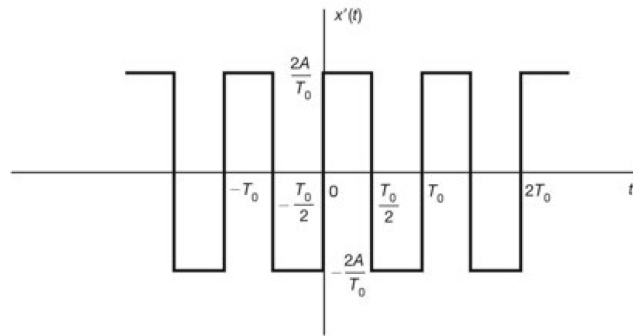
Properties of Fourier Series

1. **Linear Property:** $\alpha_1 x_1(t) + \alpha_2 x_2(t) \leftrightarrow \alpha_1 c_{1,k} + \alpha_2 c_{2,k}$
2. **Time Shifting:** $x(t-t_0) \leftrightarrow e^{-jk\omega_0 t_0} c_k$
3. **Conjugate Property:** $x^*(t) \leftrightarrow c_{-k}^*$
4. **Frequency Shifting:** $e^{jk_0 \omega_0 t} x(t) \leftrightarrow c_{k-k_0}$
5. **Time Reversal Property:** $x(-t) \leftrightarrow c_{-k}$
6. **Time Scaling Property:** $x(\alpha t) \leftrightarrow c_k$ with period $\frac{T_0}{\alpha}$
7. **Differentiation:** $\frac{dx(t)}{dt} \leftrightarrow jk\omega_0 c_k$
8. **Integration:** $\int x(t) dt \leftrightarrow \frac{c_k}{jk\omega_0}$
9. **Periodic Convolution:** $x_1(t) \otimes x_2(t) = \int_{T_0} x_1(\tau) x_2(t-\tau) d\tau \leftrightarrow T_0 c_{1,k} c_{2,k}$
10. **Multiplication Property:** $x_1(t)x_2(t) \leftrightarrow \sum_{l=-\infty}^{\infty} c_{1,l} c_{2,k-l}$

[Example 3-6] Derive the complex exponential FS coefficients of the following signal using the differentiation / or integration property. Refer [Schaum's text, Problem 5.9]



Solution) The differentiation of the given signal is plotted below, which is similar to the signal in Example 3-2. Case (c). The amplitude of Ex 3-2. Case (c) was A , whereas the that of the illustrated signal is $\frac{2A}{T_0}$.



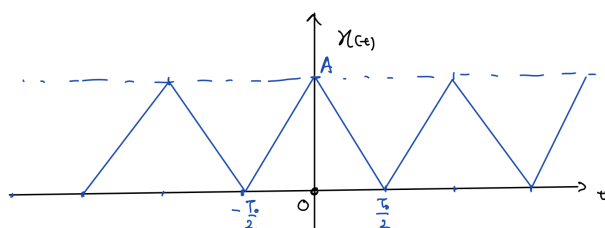
By using the integration property, the complex exponential FS coefficients of the given signal is $c_k = \frac{c_{c,k}}{jk\omega_0}$ for $k \neq 0$, where $c_{c,k}$ represents the FS coefficients of Ex 3-2. Case (c). The constant c_0 for $k = 0$ can not determined by the integration property, so it should be derived through the following equation

$$c_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt = \frac{1}{T_0} \cdot \frac{AT_0}{2} = \frac{A}{2},$$

$$c_k = \begin{cases} 0 & \text{for even } k = 2m, \\ \frac{4A}{(jk)^2 \omega_0 T_0} = -\frac{2A}{\pi^2 (2m+1)^2} & \text{for odd } k = 2m+1 \end{cases} \quad (3.24)$$



[Example 3-7] Derive the complex exponential FS coefficients of the following signal using the property of FS.



$$x(t) = \begin{cases} A \left(1 - \frac{2}{T_0} t \right) & \text{for } 0 \leq t < \frac{T_0}{2} \\ A \left(1 + \frac{2}{T_0} t \right) & \text{for } -\frac{T_0}{2} \leq t < 0. \end{cases}$$

$$x(t + T_0) = x(t)$$

Solution) The signal is a time-shifted version of Example 3-6, i.e., $x(t) = x\left(t + \frac{T_0}{2}\right)$. Due to the time shifting property, the FS coefficients are determined by $e^{jk\omega_0 T_0/2} c_k = (-1)^k c_k$.

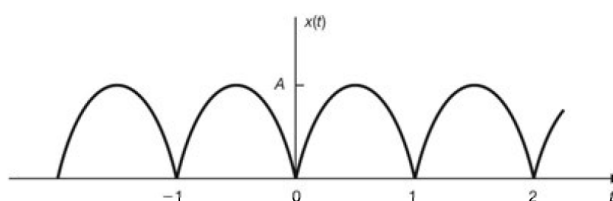
$$\begin{aligned} c_0 &= \frac{A}{2}, \\ c_k &= \begin{cases} 0 & \text{for even } k = 2m, \\ \frac{(-1)^{k+1} 2A}{(k\pi)^2} = \frac{2A}{\pi^2 (2m+1)^2} & \text{for odd } k = 2m+1 \end{cases} \end{aligned} \quad (3.25)$$



[Example 3-8] Consider a signal defined by $x(t) = |A \sin(\pi t)|$. Refer [Schaum's text, Problem 5.61]

- Sketch $x(t)$ and find its fundamental period T_0 and angular frequency ω_0
- Find the complex exponential FS series and the trigonometric FS series of $x(t)$

Solution)



The fundamental period and fundamental angular frequency of the given signal are

$$T_0 = 1, \quad \omega_0 = \frac{2\pi}{T_0} = 2\pi \quad (3.26)$$

The complex exponential FS coefficients can be derived as follows

$$\begin{aligned} c_0 &= A \int_0^1 \sin(\pi t) dt = \frac{A}{\pi} \cos(\pi t) \Big|_1^0 = \frac{2A}{\pi}, \\ c_k &= A \int_0^1 \sin(\pi t) e^{jk2\pi t} dt = \frac{2A}{\pi} \left(\frac{1}{1-4k^2} \right), \end{aligned} \quad (3.27)$$

where the integral $\int_0^1 \sin(\pi t) e^{jk2\pi t} dt$ can be derived by using integration by parts two times

$$\int_0^1 \sin(\pi t) e^{jk2\pi t} dt = \frac{2}{\pi} \left(\frac{1}{1-4k^2} \right). \quad (3.28)$$

The trigonometric Fourier coefficients can be derived as follows.

$$a_0 = 2c_0 = \frac{4A}{\pi}, \quad a_k = 2 \operatorname{Re} [c_k] = \frac{4A}{\pi} \left(\frac{1}{1-4k^2} \right), \quad b_k = -2 \operatorname{Im} [c_k] = 0 \quad (3.29)$$



Integral	Expression
$\int e^{-j\omega t} dt$	$\frac{1}{-j\omega} e^{-j\omega t}$
$\int t e^{-j\omega t} dt$	$(\frac{t}{-j\omega} - \frac{1}{(-j\omega)^2}) e^{-j\omega t}$
$\int \sin(at) dt$	$-\frac{1}{a} \cos(at)$
$\int \cos(at) dt$	$\frac{1}{a} \sin(at)$

Integral	Expression
$\int t \sin(at) dt$	$\frac{1}{a^2} \sin(at) - \frac{t}{a} \cos(at)$
$\int t \cos(at) dt$	$\frac{1}{a^2} \cos(at) + \frac{t}{a} \sin(at)$
$\int \sin(\omega_0 t) e^{at} dt$	$\frac{a \sin(\omega_0 t) - \omega_0 \cos(\omega_0 t)}{a^2 + \omega_0^2} e^{at}$
$\int \cos(\omega_0 t) e^{at} dt$	$\frac{a \cos(\omega_0 t) + \omega_0 \sin(\omega_0 t)}{a^2 + \omega_0^2} e^{at}$

Half-rectified sine wave	$x(t) = \begin{cases} A \sin(\frac{2\pi}{T_0} t) & 0 \leq t < T_0/2 \\ 0 & -\frac{T_0}{2} \leq t < 0 \end{cases}$	$a_n = \begin{cases} \frac{A}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd and } \neq \pm 1 \\ -j0.25nA & n = \pm 1 \end{cases}$
Full-rectified sine wave	$x(t) = \begin{cases} A \sin(\frac{2\pi}{T_0} t) & 0 \leq t < T_0/2 \\ -A \sin(\frac{2\pi}{T_0} t) & -\frac{T_0}{2} \leq t < 0 \end{cases}$	$a_n = \begin{cases} \frac{2A}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$
Pulse train	$x(t) = \begin{cases} A & -\tau/2 \leq t < \tau/2 \\ 0 & \text{otherwise} \end{cases}, \tau < T_0$	$a_0 = \frac{A\tau}{T_0}, a_n = \frac{A\tau \sin(n\omega_0\tau/2)}{T_0 n\omega_0\tau/2}$
Triangular wave	$x(t) = \begin{cases} -\frac{2A}{\tau}(t - \frac{\tau}{2}) & 0 \leq t < \frac{\tau}{2} \\ \frac{2A}{\tau}(t + \frac{\tau}{2}) & -\frac{\tau}{2} \leq t < 0 \\ 0 & \text{otherwise} \end{cases}, \tau < T_0$	$a_0 = \frac{A\tau}{2T_0}, a_n = \frac{8A \sin^2(n\omega_0\tau/4)}{T_0\tau (n\omega_0)^2} = \frac{4A}{\pi n^2\omega_0\tau} \sin^2(\frac{n\omega_0\tau}{4})$
Sawtooth train	$x(t) = \begin{cases} \frac{A}{\tau}(t + \frac{\tau}{2}) & -\frac{\tau}{2} \leq t < \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}, \tau < T_0$	$a_0 = \frac{A\tau}{2T_0}, a_n = \frac{A}{T_0} \left[\frac{j}{n\omega_0} e^{-j\frac{n\omega_0\tau}{2}} - j\frac{\tau}{\tau(n\omega_0)^2} \sin(\frac{n\omega_0\tau}{2}) \right]$

4 Continuous Time Fourier Transform

Major References:

- Chapter 4, *Signals and Systems* by Alan V. Oppenheim et. al., 2nd edition, Prentice Hall
- Chapter 5, *Schaum's Outline of Signals and Systems*, 2nd Edition, 2010, McGraw-Hill

Develope Fourier Transform from Fourier Series

Let's consider a nonperiodic signal with finite duration $x(t)$, where $x(t) = 0$ for $|t| > T_1$ and assume that $T_1 < \frac{T_0}{2}$. We introduce a periodic signal $x_{T_0}(t)$ by repeating $x(t)$ with fundamental period T_0 as follows

$$x_{T_0}(t + T_0) = x_{T_0}(t) \quad \text{where} \quad x_{T_0}(t) = x(t) \quad \text{for} \quad -\frac{T_0}{2} \leq t < \frac{T_0}{2}$$



If we let $T_0 \rightarrow \infty$, we have the following results by using the complex exponential FS representation

$$\begin{aligned} x(t) &= \lim_{T_0 \rightarrow \infty} x_{T_0}(t) = \lim_{T_0 \rightarrow \infty} \sum_{k=-\infty}^{\infty} c_k e^{j2\pi \frac{k}{T_0} t} = \lim_{T_0 \rightarrow \infty} \sum_{k=-\infty}^{\infty} \left\{ \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_{T_0}(\tau) e^{-j2\pi \frac{k}{T_0} \tau} d\tau \right\} e^{j2\pi \frac{k}{T_0} t} \\ &= \lim_{T_0 \rightarrow \infty} \sum_{k=-\infty}^{\infty} \left\{ \frac{1}{T_0} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi \frac{k}{T_0} \tau} d\tau \right\} e^{j2\pi \frac{k}{T_0} t} \end{aligned} \quad (4.1)$$

Since $x_{T_0}(t) = x(t)$ for $-\frac{T_0}{2} \leq t < \frac{T_0}{2}$ and $x(t) = 0$ for $|t| > \frac{T_0}{2}$, the last equality of (4.1) follows

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_{T_0}(\tau) e^{-jk\omega_0 \tau} d\tau = \int_{-\infty}^{\infty} x(\tau) e^{-jk\omega_0 \tau} d\tau$$

Let's denote $f = \frac{k}{T_0}$, then $\lim_{T_0 \rightarrow \infty} f = \Delta f = \frac{\Delta k}{T_0}$. Since k is an integer, $\Delta k = 1$. By applying $T_0 \Delta f = 1$ to (4.1),

$$\begin{aligned} x(t) &= \lim_{T_0 \rightarrow \infty} \sum_{k=-\infty}^{\infty} \left\{ \frac{1}{T_0} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi \frac{k}{T_0} \tau} d\tau \right\} e^{j2\pi \frac{k}{T_0} t} \cdot 1 = \lim_{T_0 \rightarrow \infty} \sum_{k=-\infty}^{\infty} \underbrace{\left\{ \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} d\tau \right\}}_{\triangleq X(f)} e^{j2\pi f t} \cdot \Delta f \\ &= \lim_{T_0 \rightarrow \infty} \sum_{T_0 f = -\infty}^{\infty} X(f) e^{j2\pi f t} \cdot \Delta f = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df, \end{aligned} \quad (4.2)$$

where we denoted $X(f) \triangleq \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$ and applied the definition of the Riemann Integral.

4.1 Fourier Transform Pair and Properties

Based on (4.1, 4.2), we derived the Fourier Transform pair of an aperiodic continuous time signal $x(t)$ as follows.

1. CT - Fourier Transform Pair

For a non-periodic CT signal $x(t)$, the FT pair **in terms of frequency f** are given by

- Fourier Transform of $x(t)$; $\mathcal{F}\{x(t)\} = X(f)$
- Inverse Fourier Transform of $X(f)$; $\mathcal{F}^{-1}\{X(f)\} = x(t)$
- Notation; $x(t) \xleftrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} X(f)$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \quad \xleftrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} \quad X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \quad (4.3)$$

The FT pair **in terms of angular frequency $\omega = 2\pi f$** are given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad \xleftrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} \quad X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad (4.4)$$

2. Dirichlet Condition (Sufficient conditions for FT to exist)

1. $x(t)$ is absolutely integrable $\int_{-\infty}^{\infty} |x(t)| dt < \infty$
2. $x(t)$ has a finite number of maxima and minima within any finite interval of t .
3. $x(t)$ has a finite number of discontinuities within any finite interval of t , and each of these discontinuities is finite.

If $x(t)$ satisfies the Dirichlet condition, then the corresponding Fourier transform is convergent.

✱ Properties of Fourier Transform

1. Linear Property

If the FT of $x_1(t)$ and $x_2(t)$ are $X_1(f)$ and $X_2(f)$, then the FT of $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ is given by

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \leftrightarrow \alpha_1 X_1(f) + \alpha_2 X_2(f)$$

Proof) The FT of the linearly combined signals $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ is

$$\begin{aligned} \int_{-\infty}^{\infty} [\alpha_1 x_1(t) + \alpha_2 x_2(t)] e^{-j2\pi ft} dt &= \alpha_1 \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi ft} dt + \alpha_2 \int_{-\infty}^{\infty} x_2(t) e^{-j2\pi ft} dt \\ &= \alpha_1 X_1(f) + \alpha_2 X_2(f). \end{aligned}$$

2. Parameter Shifting

If the FT of $x(t)$ is $X(f)$, then the FT of the following mapped signals are given by

$$\mathcal{F}[x(t - t_0)] = e^{-j2\pi f t_0} X(f), \quad \mathcal{F}[x(t) e^{\pm j2\pi f_0 t}] = X(f \mp f_0)$$

Proof) The FT of $x(t - t_0)$ and $x(t) e^{\pm j2\pi f_0 t}$ are derived as below

$$\begin{aligned} \int_{-\infty}^{\infty} x(t - t_0) e^{-j2\pi ft} dt &= \int_{-\infty}^{\infty} x(v) e^{-j2\pi f v} dv \cdot e^{-j2\pi f t_0} = X(f) e^{-j2\pi f t_0}, \\ \int_{-\infty}^{\infty} x(t) e^{\pm j2\pi f_0 t} e^{-j2\pi ft} dt &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi (f \mp f_0) t} dt = X(f \mp f_0), \end{aligned}$$

Ex 4-3)

a) $\mathcal{F}\left(\text{rect}\left(\frac{t}{3}\right)\right)$

$$X(f) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{3}\right) e^{-j2\pi ft} dt$$

$$= \int_{-\frac{3}{2}}^{\frac{3}{2}} e^{-j2\pi ft} dt = \frac{e^{-j2\pi ft}}{(-j2\pi f)} \Big|_{-\frac{3}{2}}^{\frac{3}{2}}$$

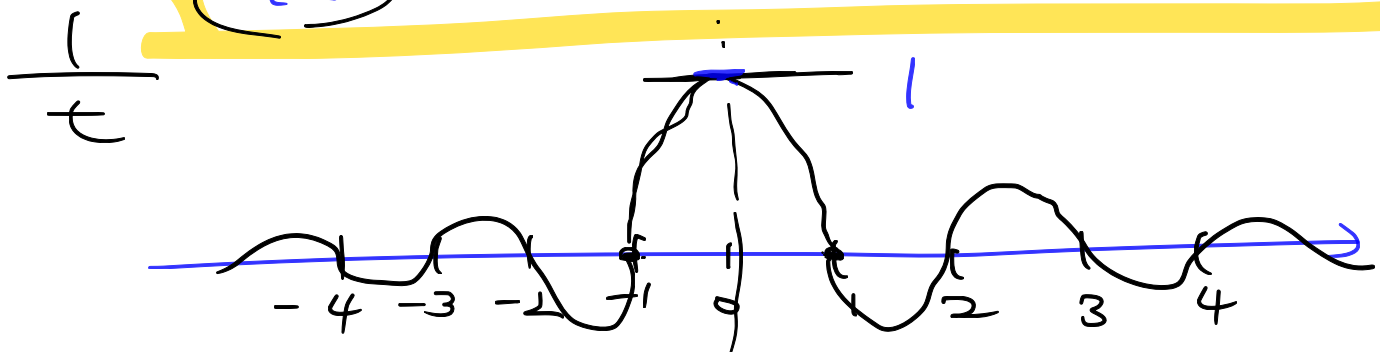
$$= \frac{1}{j2\pi f} \left[e^{j\pi f 3} - e^{-j\pi f 3} \right]$$

$e^{j\theta} = \cos \theta + j \sin \theta$

$$= \frac{\sin(\pi f 3) \cdot 3}{\pi f \cdot 3} = 3 \text{sinc}(f \cdot 3)$$

$\pi t = n\pi \rightarrow$ if $t = n$, where n is an integer

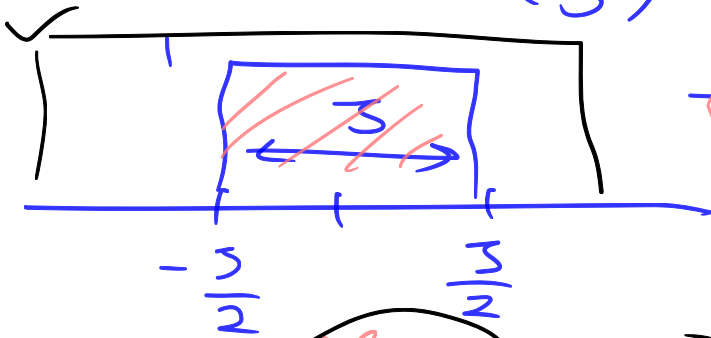
$\frac{\sin(\pi t)}{\pi t} = \text{sinc}(t)$ function



$$t \rightarrow 0 \quad \lim_{t \rightarrow 0} \text{Sinc}(t) = \frac{\sin(\pi t)}{\pi t} \rightarrow \frac{0}{0} = 1$$

$$\lim_{t \rightarrow 0} \text{Sinc}(t) = \frac{\pi \cos(\pi t)}{\pi} \lim_{t \rightarrow 0} (t \rightarrow 0) = \frac{0}{0} = 1$$

$$\text{rect}\left(\frac{t}{\tau}\right)$$

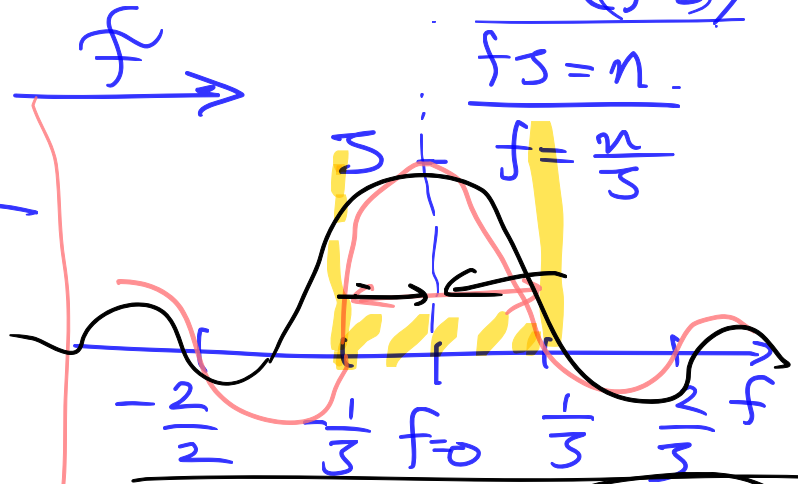


$x(t)$
Width = τ

time

$$\tau \cdot \text{Sinc}(f\tau)$$

$$f\tau = n \Rightarrow f = \frac{n}{\tau}$$



$X(f)$
 $\tau \Rightarrow$

Frequency

$$\tau = \frac{1}{f_{\text{freq}}}$$

Properties of FT.

$$\begin{aligned} \mathcal{F}\{x_1(t)\} &= X_1(f) \\ \mathcal{F}\{x_2(t)\} &= X_2(f) \end{aligned}$$

1. linearity

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \xrightarrow{\mathcal{F}} \alpha_1 X_1(f) + \alpha_2 X_2(f)$$

Proof)

$$\begin{aligned} & \int_{-\infty}^{\infty} (\alpha_1 x_1(t) + \alpha_2 x_2(t)) e^{-j2\pi f t} dt \\ &= \alpha_1 \underbrace{\int_{-\infty}^{\infty} x_1(t) e^{-j2\pi f t} dt}_{X_1} + \alpha_2 \underbrace{\int_{-\infty}^{\infty} x_2(t) e^{-j2\pi f t} dt}_{X_2} \end{aligned}$$

2. Parameter Shifting

$$\begin{aligned} x(t-t_0) &\longleftrightarrow X(f) e^{-j2\pi f t_0} \\ x(t) e^{+j2\pi f_0 t} &\longleftrightarrow X(f-f_0) \end{aligned}$$

$$\begin{aligned} & \int_{-\infty}^{\infty} x(t-t_0) e^{-j2\pi f t} dt \\ & \quad \begin{aligned} & t-t_0 = \tau \\ & \boxed{t = \tau + t_0} \\ & dt = d\tau \end{aligned} \\ &= \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f (\tau + t_0)} d\tau \rightarrow X(f) \times e^{-j2\pi f t_0} \end{aligned}$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

3. Scaling $\mathcal{F}\{x(\alpha t)\} = X(-f)$

$$x(\alpha t) \xrightarrow{\mathcal{F}} \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right)$$

Proof)

$$\int_{-\infty}^{\infty} x(\alpha t) e^{-j2\pi ft} dt$$

$$t = \frac{\tau}{\alpha}$$

$$\alpha dt = d\tau$$

$$\frac{1}{\alpha} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi \frac{f}{\alpha} \tau} d\tau \Rightarrow \frac{1}{\alpha} X\left(\frac{f}{\alpha}\right)$$

if $\alpha > 0$

$$\frac{1}{\alpha} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi \frac{f}{\alpha} \tau} d\tau$$

$$= -\frac{1}{\alpha} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi \frac{f}{\alpha} \tau} d\tau$$

$$= -\frac{1}{\alpha} X\left(\frac{f}{\alpha}\right) \text{ if } \alpha < 0$$

4. Conjugate Property

$$x(t) \xrightarrow{F} X(f)$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{j2\pi ft} dt$$

$$X(-f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$F(x(t)) = X(f)$$

where we used a change of variable ($t - t_0 = v$) in the first equality of the first expression.

3. Scaling

If the FT of $x(t)$ is $X(f)$, then the FT of the following scaled signal is given by

$$\mathcal{F}[x(at)] = \frac{1}{|a|} X\left(\frac{f}{a}\right), \quad \mathcal{F}[x(-t)] = X(-f)$$

Proof) The FT of $x(at)$ is given by

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt = \begin{cases} \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \frac{\tau}{a}} d\tau, & \text{if } a > 0 \\ -\frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \frac{\tau}{a}} d\tau, & \text{if } a < 0 \end{cases}$$

4. Conjugate Property

If the FT of $x(t)$ is $X(f)$, the FT of $x^*(t)$ for arbitrary signal $x(t)$ is given by

$$\mathcal{F}[x^*(t)] = X^*(-f).$$

If $x(t)$ is a real-valued signal, then $X^*(-f) = X(f)$ follows by the conjugate property.

Proof) By taking the complex conjugate of (4.3), we get

$$X^*(f) = \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \right]^* = \int_{-\infty}^{\infty} x^*(t) e^{j2\pi ft} dt \Rightarrow X^*(-f) = \int_{-\infty}^{\infty} x^*(t) e^{-j2\pi ft} dt$$

5. Duality

If the FT of $x(t)$ is $X(f)$, the FT of $X(t)$ can be derived as follows

$$\mathcal{F}[X(t)] = x(-f)$$

Proof) By using a change of variable ($f' \leftarrow -t, t' \leftarrow f$), we get

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \Rightarrow x(-f') = \int_{-\infty}^{\infty} X(t') e^{-j2\pi f' t'} dt'$$

6. Convolution and Multiplication

If the FT of $x(t)$ and $y(t)$ are $X(f)$ and $Y(f)$, respectively, the FT of the following signals are given by

$$\mathcal{F}[x(t) * y(t)] = X(f)Y(f), \quad \mathcal{F}[x(t)y(t)] = X(f) * Y(f)$$

Proof) The FT of the convolution signal can be derived as

$$\begin{aligned} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau \right] e^{-j2\pi ft} dt &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} y(t - \tau) e^{-j2\pi f(t - \tau)} dt d\tau \\ &= \left[\int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau \right] \times \left[\int_{-\infty}^{\infty} y(l) e^{-j2\pi fl} dl \right] = X(f)Y(f) \end{aligned}$$

where we used a change of variable ($t - \tau = l$) in the second equality. Similarly,

$$\begin{aligned} \mathcal{F}^{-1}[X(f) * Y(f)] &= \int_{-\infty}^{\infty} \{X(f) * Y(f)\} e^{j2\pi ft} df = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} X(l) Y(f - l) dl \right\} e^{j2\pi ft} df \\ &= \left\{ \int_{-\infty}^{\infty} X(l) e^{j2\pi lt} dl \right\} \times \left\{ \int_{-\infty}^{\infty} Y(f - l) e^{j2\pi(f-l)t} df \right\} = x(t)y(t) \end{aligned}$$

where we used a change of variable ($f - l = f'$) in the last equality.

7. Differentiation

If the FT of $x(t)$ is $X(f)$, the FT of the following signals are given by

$$\mathcal{F} \left[\frac{d^n x(t)}{dt^n} \right] = (j2\pi f)^n X(f), \quad \mathcal{F} [(-j2\pi t)^n x(t)] = \frac{d^n X(f)}{df^n}$$

Proof) By differentiating (4.3), we get

$$\begin{aligned} \frac{d^n x(t)}{dt^n} &= \int_{-\infty}^{\infty} (j2\pi f)^n X(f) e^{j2\pi f t} df = \mathcal{F}^{-1} [(j2\pi f)^n X(f)], \\ \frac{d^n X(f)}{df^n} &= \int_{-\infty}^{\infty} (-j2\pi t)^n x(t) e^{-j2\pi f t} dt = \mathcal{F} [(-j2\pi t)^n x(t)] \end{aligned}$$

8. Integration

If the FT of $x(t)$ is $X(f)$, the FT of the following signal is given by

$$\mathcal{F} \left[\int_{-\infty}^t x(\tau) d\tau \right] = \frac{1}{2} X(0) \delta(f) + \frac{1}{j2\pi f} X(f)$$

Proof) The integral can be expressed in terms of convolution as follows

$$\int_{-\infty}^t x(\tau) d\tau = x(t) * u(t)$$

Then, by using the convolution property of the FT, we get

$$\mathcal{F} \left[\int_{-\infty}^t x(\tau) d\tau \right] = \mathcal{F} [x(t) * u(t)] = X(f) \times \mathcal{F} [u(t)]$$

where the FT pair of the unit step function is derived in the example problem

$$\mathcal{F} [u(t)] = \frac{1}{2} \delta(f) + \frac{1}{j2\pi f}$$

Hence, we obtained the FT of integrated signals as follows

$$\mathcal{F} [x(t) * u(t)] = X(f) \times \mathcal{F} [u(t)] = \frac{1}{2} X(0) \delta(f) + \frac{1}{j2\pi f} X(f)$$

9. Parseval's Theorem

If the FT of $x(t)$ is $X(f)$, the Parseval's theorem is

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Property	Signal	Fourier Transform
	$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$ $y(t) = \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft}df$	$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$ $Y(f) = \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt$
Linearity	$Ax(t) + By(t)$	$AX(f) + BY(f)$
Time Shift	$x(t - t_0)$	$e^{-j2\pi ft_0} X(f)$
Frequency Shift	$e^{j2\pi f_0 t} x(t)$	$X(f - f_0)$
Scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{f}{a}\right)$
Duality	$X(t)$	$x(-f)$
Complex Conjugate	$x^*(t)$	$X^*(-f)$
Area	$X(0) = \int_{-\infty}^{\infty} x(t)dt$	
	$x(0) = \int_{-\infty}^{\infty} X(f)df$	
Time Differentiation	$x'(t)$	$j2\pi f X(f)$
	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
Freq. Differentiation	$-j2\pi t x(t)$	$X'(f)$
	$(-j2\pi t)^n x(t)$	$\frac{d^n X(f)}{df^n}$
Time Integration	$\int_{-\infty}^t x(\tau)d\tau$	$\frac{1}{j2\pi f} X(f)$, for $X(0) = 0$
Convolution	$\int_{-\infty}^{\infty} x(\tau)y(t - \tau)d\tau$	$X(f)Y(f)$
Multiplication	$x(t)y(t)$	$\int_{-\infty}^{\infty} X(\nu)Y(f - \nu)d\nu$
Energy Conservation (Parseval's Theorem)	$\int_{-\infty}^{\infty} x(t) ^2 dt = \int_{-\infty}^{\infty} X(f) ^2 df$	

Figure 4.1: Properties of the Fourier Transform

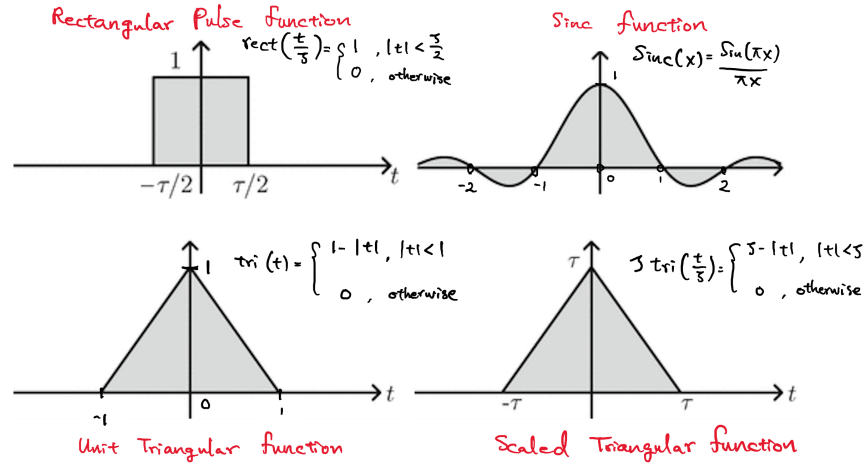
	$g(t)$	$G(f)$	
✓	$e^{-at}u(t)$	$\frac{1}{a + j2\pi f}$	$a > 0$
✓	2 $e^{at}u(-t)$	$\frac{1}{a - j2\pi f}$	$a > 0$
	3 $e^{-a t }$	$\frac{2a}{a^2 + (2\pi f)^2}$	
	4 $te^{-at}u(t)$	$\frac{1}{(a + j2\pi f)^2}$	$a > 0$
	5 $t^n e^{-at}u(t)$	$\frac{n!}{(a + j2\pi f)^{n+1}}$	
	6 $\delta(t)$	1	
	7 1	$\delta(f)$	
	8 $e^{j2\pi f_0 t}$	$\delta(f - f_0)$	
	9 $\cos 2\pi f_0 t$	$0.5 [\delta(f + f_0) + \delta(f - f_0)]$	
	10 $\sin 2\pi f_0 t$	$j0.5 [\delta(f + f_0) - \delta(f - f_0)]$	
	11 $u(t)$	$\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$	
	12 $\text{sgn } t$	$\frac{2}{j2\pi f}$	
	$\cos 2\pi f_0 t u(t)$	$\frac{1}{4}[\delta(f - f_0) + \delta(f + f_0)] + \frac{j2\pi f}{(2\pi f_0)^2 - (2\pi f)^2}$	
	14 $\sin 2\pi f_0 t u(t)$	$\frac{1}{4j}[\delta(f - f_0) - \delta(f + f_0)] + \frac{2\pi f_0}{(2\pi f_0)^2 - (2\pi f)^2}$	
	15 $e^{-at} \sin 2\pi f_0 t u(t)$	$\frac{2\pi f_0}{(a + j2\pi f)^2 + 4\pi^2 f_0^2}$	
	16 $e^{-at} \cos 2\pi f_0 t u(t)$	$\frac{a + j2\pi f}{(a + j2\pi f)^2 + 4\pi^2 f_0^2}$	
✓	18 $\text{rect}\left(\frac{t}{\tau}\right)$	$\tau \cdot \text{sinc}(f\tau)$	
	$\text{sinc}(2Bt)$	$\frac{1}{2B} \text{rect}\left(\frac{f}{2B}\right)$	
	19 $\tau \text{tri}\left(\frac{t}{\tau}\right)$	$\tau^2 \text{sinc}^2(f\tau)$	
	20 $\text{sinc}^2(2Bt)$	$\frac{1}{2B} \text{tri}\left(\frac{f}{2B}\right)$	
	21 $\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{1}{f_0} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$	
	22 $e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-2(\sigma\pi f)^2}$	

Figure 4.2: Table of the Fourier Transform

[Example 4-3. Functions related to Rectangular Pulse] Derive the FT of the following signals where $\tau > 0$

- a) $\mathcal{F} \left[\text{rect} \left(\frac{t}{\tau} \right) \right] =$ b) $\mathcal{F} [\text{sinc} (2Bt)] =$
 c) $\mathcal{F} \left[\tau \text{tri} \left(\frac{t}{\tau} \right) \right] =$ d) $\mathcal{F} [\text{sinc}^2 (2Bt)] =$
 e) $\mathcal{F} \left[\text{rect} \left(\frac{t-t_0}{\tau} \right) \right] =$ f) $\mathcal{F} [\text{sinc} (2B(t-t_0))] =$

where the rectangular pulse $\text{rect}(t)$, Sinc function $\text{sinc}(t)$, and triangular function $\text{tri}(t)$ are illustrated below.



Solution) The FT of the $\text{rect}(t)$ in (a) can be directly derived using (4.3)

$$(a) \mathcal{F} \left[\text{rect} \left(\frac{t}{\tau} \right) \right] = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j2\pi f t} dt = \frac{1}{j2\pi f} e^{-j2\pi f t} \Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{2j \sin(\pi f \tau)}{j2\pi f} = \tau \cdot \text{sinc}(f\tau).$$

The y-intercept of the Sinc function is 1 and the zero-crossing occurs at every integer t , so $\tau \text{sinc}(f\tau)$ has a maximum value τ at $f = 0$ with zero-crossing at $f = \frac{n}{\tau}$ for any nonzero integer n .

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \Rightarrow 0 \quad \text{for } t = \text{arbitrary nonzero integer}, \quad \lim_{t \rightarrow 0} [\text{sinc}(t)] = 1$$

The FT of (b) can be derived by using the duality property with (a) as follows

$$(b) \frac{1}{\tau} \text{rect} \left(\frac{t}{\tau} \right) \xleftrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} \text{sinc}(f\tau), \quad \xrightarrow{\text{Duality}} \text{sinc}(2Bt) \xleftrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} \frac{1}{2B} \text{rect} \left(\frac{f}{2B} \right)$$

where we substituted $\tau = 2B$ and used a change of variable ($t \rightarrow -f'$, $f \rightarrow t'$). Hence, $\mathcal{F} [\text{sinc}(2Bt)] = \frac{1}{2B} \text{rect} \left(\frac{f}{2B} \right)$. In Example 2-1. (b) (pp26), we proved that the convolution of two rectangular pulse signal is a triangular function

$$\text{rect} \left(\frac{t}{\tau} \right) * \text{rect} \left(\frac{t}{\tau} \right) = \tau \cdot \text{tri} \left(\frac{t}{\tau} \right) = \begin{cases} \tau - |t| & \text{for } |t| \leq \tau \\ 0 & \text{for } |t| > \tau \end{cases}$$

Then, the FT of (c) can be obtained by convolution properties and (d) can be derived based on the duality with (c).

$$(c) \mathcal{F} \left[\tau \text{tri} \left(\frac{t}{\tau} \right) \right] = \mathcal{F} \left[\text{rect} \left(\frac{t}{\tau} \right) * \text{rect} \left(\frac{t}{\tau} \right) \right] = \mathcal{F} \left[\text{rect} \left(\frac{t}{\tau} \right) \right] \times \mathcal{F} \left[\text{rect} \left(\frac{t}{\tau} \right) \right] = \tau^2 \text{sinc}^2(f\tau),$$

$$(d) \frac{1}{\tau} \text{tri} \left(\frac{t}{\tau} \right) \xleftrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} \text{sinc}^2(f\tau), \quad \xrightarrow{\text{Duality}} \text{sinc}^2(2Bt) \xleftrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} \frac{1}{2B} \text{tri} \left(\frac{f}{2B} \right)$$

where we substituted $\tau = 2B$ and used a change of variable ($t \rightarrow -f'$, $f \rightarrow t'$). Hence, $\mathcal{F} [\text{sinc}^2 (2Bt)] = \frac{1}{2B} \text{tri} \left(\frac{f}{2B} \right)$. The FT of (e) and (f) can be obtained by using the time-shift property of (a) and frequency-shift of (b), respectively.

$$(e) \mathcal{F} \left[\text{rect} \left(\frac{t - t_0}{\tau} \right) \right] = \tau e^{-j2\pi f t_0} \text{sinc} (f\tau), \quad (f) \mathcal{F} [\text{sinc} (2B(t - t_0))] = \frac{1}{2B} e^{-j2\pi f t_0} \text{rect} \left(\frac{t}{2B} \right)$$



[Example 4-4. Functions related to Modulation] Derive the FT of the following signals where $\alpha > 0$

- | | |
|---|---|
| a) $\mathcal{F} [\cos (2\pi f_0 t)] =$ | b) $\mathcal{F} [\sin (2\pi f_0 t)] =$ |
| c) $\mathcal{F} [x(t) \cos (2\pi f_0 t)] =$ | d) $\mathcal{F} [x(t) \sin (2\pi f_0 t)] =$ |
| e) $\mathcal{F} [\cos (2\pi f_0 t) u(t)] =$ | f) $\mathcal{F} [\sin (2\pi f_0 t) u(t)] =$ |
| g) $\mathcal{F} [e^{-\alpha t} \cos (2\pi f_0 t) u(t)] =$ | h) $\mathcal{F} [e^{-\alpha t} \sin (2\pi f_0 t) u(t)] =$ |

Solution The FT of (a) and (b) can be derived by using Example 4-1 (d) and linear property as follows

$$(a) \mathcal{F} [\cos (2\pi f_0 t)] = \mathcal{F} \left[\frac{1}{2} (e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) \right] = \frac{1}{2} [\delta (f - f_0) + \delta (f + f_0)],$$

$$(b) \mathcal{F} [\sin (2\pi f_0 t)] = \mathcal{F} \left[\frac{1}{2j} (e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}) \right] = -\frac{j}{2} [\delta (f - f_0) - \delta (f + f_0)]$$

The FT of (c) and (d) can be obtained based on the multiplication property of the FT with (a) and (b).

$$(c) \mathcal{F} [x(t) \cos (2\pi f_0 t)] = X(f) * \frac{1}{2} [\delta (f - f_0) + \delta (f + f_0)] = \frac{1}{2} [X(f - f_0) + X(f + f_0)],$$

$$(d) \mathcal{F} [x(t) \sin (2\pi f_0 t)] = X(f) * \frac{1}{2j} [\delta (f - f_0) - \delta (f + f_0)] = -\frac{j}{2} [X(f - f_0) - X(f + f_0)]$$

The FT of (e) and (f) can be obtained by using Example 4-2, i.e., $\mathcal{F} [u(t)] = \frac{1}{2} \delta(f) + \frac{1}{j2\pi f}$

$$(e) \mathcal{F} [\cos (2\pi f_0 t) u(t)] = \frac{1}{2} \left[\frac{1}{2} \delta (f - f_0) + \frac{1}{j2\pi (f - f_0)} + \frac{1}{2} \delta (f + f_0) + \frac{1}{j2\pi (f + f_0)} \right]$$

$$= \frac{1}{4} [\delta (f - f_0) + \delta (f + f_0)] + \frac{j2\pi f}{(2\pi f_0)^2 - (2\pi f)^2},$$

$$(f) \mathcal{F} [\sin (2\pi f_0 t) u(t)] = \frac{1}{2j} \left[\frac{1}{2} \delta (f - f_0) + \frac{1}{j2\pi (f - f_0)} - \frac{1}{2} \delta (f + f_0) - \frac{1}{j2\pi (f + f_0)} \right]$$

$$= \frac{1}{4j} [\delta (f - f_0) - \delta (f + f_0)] + \frac{2\pi f_0}{(2\pi f_0)^2 - (2\pi f)^2}$$

The FT of (g) and (h) can be obtained by the multiplication property with $\mathcal{F} [e^{-\alpha t} u(t)] = (\alpha + j2\pi f)^{-1}$

$$(g) \mathcal{F} [e^{-\alpha t} \cos (2\pi f_0 t) u(t)] = \frac{1}{\alpha + j2\pi f} * \frac{1}{2} [\delta (f - f_0) + \delta (f + f_0)]$$

$$= \frac{1}{2} \left[\frac{1}{\alpha + j2\pi (f - f_0)} + \frac{1}{\alpha + j2\pi (f + f_0)} \right] = \frac{\alpha + j2\pi f}{(\alpha + j2\pi f)^2 + 4(\pi f_0)^2},$$

$$(h) \mathcal{F} [e^{-\alpha t} \sin (2\pi f_0 t) u(t)] = \frac{1}{\alpha + j2\pi f} * \frac{1}{2j} [\delta (f - f_0) - \delta (f + f_0)]$$

$$= \frac{1}{2j} \left[\frac{1}{\alpha + j2\pi (f - f_0)} - \frac{1}{\alpha + j2\pi (f + f_0)} \right] = \frac{2\pi f_0}{(\alpha + j2\pi f)^2 + 4(\pi f_0)^2}$$



[Example 4-5. etc] Derive the FT of the following signals where $\alpha > 0$

1. FT of a periodic signal $x(t)$ with period T_0
2. FT of the periodic impulse train $\mathcal{F} \left[\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0) \right]$
3. $\mathcal{F} [t^n e^{-\alpha t} u(t)]$

Solution In (a), we first express a periodic signal $x(t)$ using the FS, then apply $\mathcal{F} [e^{j2\pi f_0 t}] = \delta(f - f_0)$ as follows

$$(a) \mathcal{F} [x(t)] = \mathcal{F} \left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right] = \sum_{k=-\infty}^{\infty} c_k \mathcal{F} [e^{jk\omega_0 t}] = \sum_{k=-\infty}^{\infty} c_k \delta \left(f - \frac{k\omega_0}{2\pi} \right),$$

which indicates that the Fourier transform of a periodic signal consists of a sequence of equidistant impulses located at the harmonic frequencies. For (b), the FS of the periodic impulse train $\delta_{T_0}(t)$ is derived in Example 3-3, pp42

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} \Rightarrow (b) \mathcal{F} \left[\sum_{k=-\infty}^{\infty} \delta(t - kT_0) \right] = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \mathcal{F} [e^{jk\omega_0 t}] = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \delta \left(f - \frac{k\omega_0}{2\pi} \right)$$

The FT of (c) can be derived by using $\mathcal{F} [e^{-\alpha t} u(t)] = \frac{1}{\alpha + j2\pi f}$ and the higher order differentiation of the FT

$$\mathcal{F} [t^n x(t)] = \frac{1}{(-j2\pi)^n} \frac{d^n X(f)}{df^n} \text{ where } x(t) = e^{-\alpha t} u(t), X(f) = \frac{1}{\alpha + j2\pi f} \Rightarrow (c) \mathcal{F} [t^n e^{-\alpha t} u(t)] = \frac{n!}{(\alpha + j2\pi f)^{n+1}}$$

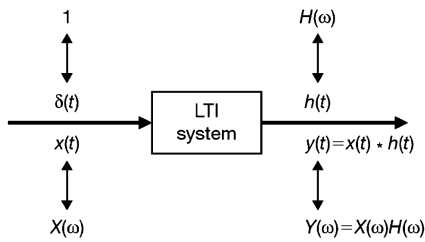


4.2 Frequency Response

Given the FT of an input $x(t)$ and the impulse response $h(t)$ as $X(f)$ and $H(f)$, the output $y(t)$ of an LTI system is

$$\begin{aligned} y(t) &= h(t) * x(t) = \mathcal{F}^{-1} (Y(f)) = \mathcal{F}^{-1} (X(f)H(f)) \\ &= \int_{-\infty}^{\infty} Y(f) e^{j2\pi f t} df = \int_{-\infty}^{\infty} X(f)H(f) e^{j2\pi f t} df \quad \text{for Aperiodic input} \end{aligned} \quad (4.11)$$

where we refer the FT of the impulse response $H(f)$ as the **Frequency Response**, the magnitude of the frequency response $|H(f)|$ as the **Magnitude Response**, the phase of the frequency response $\theta_H(f)$ as the **Phase response**.



- Frequency Response:

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t} dt = |H(f)| e^{j\theta_H(f)}$$

- Magnitude Response: $|H(f)|$
- Phase response: $\theta_H(f)$

The output in (4.11) holds for a non-periodic input signal $x(t)$. Given a periodic input $x(t)$,

- If $x(t) = e^{jk\omega_0 t}$, then $\mathcal{F} (x(t)) = \delta(f - kf_0)$, where $f_0 = \frac{\omega_0}{2\pi}$, and

$$Y(f) = H(f)X(f) = H(f)\delta(f - kf_0) = H(kf_0)\delta(f - kf_0)$$

- If $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, then $\mathcal{F}(x(t)) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_0)$ and

$$Y(f) = H(f)X(f) = \sum_{k=-\infty}^{\infty} c_k H(f) \delta(f - kf_0) = \sum_{k=-\infty}^{\infty} c_k c_k H(kf_0) \delta(f - kf_0) \quad \text{for periodic input}$$

* LTI Systems Characterized by Differential Equations

If the LTI system is described by differential equations, we can derive the Frequency response using the differentiation property of the Fourier transform as follows

$$\begin{aligned} \sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} &= \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k} \xrightarrow{\mathcal{F}} \sum_{k=0}^N a_k \mathcal{F} \left\{ \frac{d^k y(t)}{dt^k} \right\} = \sum_{k=0}^M b_k \mathcal{F} \left\{ \frac{d^k x(t)}{dt^k} \right\}, \quad M \leq N \\ \Rightarrow H(f) &= \frac{Y(f)}{X(f)} = \frac{\sum_{k=0}^M b_k (j2\pi f)^k}{\sum_{k=0}^N a_k (j2\pi f)^k} \Leftrightarrow h(t) = \mathcal{F}^{-1}(H(f)) \end{aligned}$$

[Example 4-6.] Consider CT-LTI systems described by the following differential equations. Derive the frequency response $H(f)$ and the corresponding impulse response $h(t)$. Furthermore, derive the system output when the input signal is $x_1(t) = e^{-t}u(t)$, $x_2(t) = te^{-2t}u(t)$, and $x_3(t) = u(t)$, respectively.

- $\frac{dy(t)}{dt} + 2y(t) = x(t) + \frac{dx(t)}{dt} \Rightarrow H(f), h(t), y_i(t) = \mathbb{T}\{x_i(t)\}, \text{ where } i = \{1, 2, 3\}$
- $\frac{dy(t)}{dt} + 2y(t) = x(t) \Rightarrow H(f), h(t), y_i(t) = \mathbb{T}\{x_i(t)\}, \text{ where } i = \{1, 2, 3\}$
- $\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t) \Rightarrow H(f), h(t), y_i(t) = \mathbb{T}\{x_i(t)\}, \text{ where } i = \{1, 2, 3\}$
- $\frac{d^2 y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t) \Rightarrow H(f), h(t), y_i(t) = \mathbb{T}\{x_i(t)\}, \text{ where } i = \{1, 2, 3\}$

Solution Since $\mathcal{F} \left[\frac{d^n x(t)}{dt^n} \right] = (j2\pi f)^n X(f)$, the FT of each differential equations can be expressed as

$$\begin{aligned} \text{(a)} \quad \mathcal{F} \left\{ \frac{dy(t)}{dt} + 2y(t) = x(t) + \frac{dx(t)}{dt} \right\} &\Leftrightarrow (j2\pi f) Y(f) + 2Y(f) = X(f) + (j2\pi f) X(f), \\ \Rightarrow H(f) = \frac{Y(f)}{X(f)} &= \frac{1 + j2\pi f}{2 + j2\pi f} = 1 - \frac{1}{1 + j2\pi f} \xrightarrow{\mathcal{F}^{-1}} h(t) = \delta(t) - e^{-2t}u(t), \end{aligned}$$

where we used the FT pairs in Table 4.2. The FT of each input signals are given by

$$\begin{aligned} X_1(f) &= \mathcal{F}\{e^{-t}u(t)\} = \frac{1}{1 + j2\pi f}, \quad X_2(f) = \mathcal{F}\{te^{-2t}u(t)\} = \frac{1}{(2 + j2\pi f)^2}, \\ X_3(f) &= \mathcal{F}\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f). \end{aligned}$$

Since the FT of the system output is given by $Y_i(f) = \mathcal{F}(x_i(t) * h(t)) = X_i(f)H(f)$, the output for each signals are

$$\begin{aligned} Y_1(f) &= \frac{1}{2 + j2\pi f} \Rightarrow y_1(t) = e^{-2t}u(t), \\ Y_2(f) &= \frac{1 + j2\pi f}{(2 + j2\pi f)^3} = \frac{1}{(2 + j2\pi f)^2} - \frac{1}{(2 + j2\pi f)^3} \Rightarrow y_2(t) = te^{-2t}u(t) - \frac{1}{2}t^2 e^{-2t}u(t), \\ Y_3(f) &= \frac{1}{4}\delta(f) + \frac{1 + j2\pi f}{j2\pi f(2 + j2\pi f)} = \frac{1}{2} \left[\frac{1}{2}\delta(f) + \frac{1}{j2\pi f} \right] + \frac{1}{2} \left(\frac{1}{2 + j2\pi f} \right) \Rightarrow y_3(t) = \frac{1}{2}u(t) + \frac{1}{2}e^{-2t}u(t), \end{aligned}$$

where we applied the [Partial Fraction Expansion](https://bit.ly/3cSxtCi). Refer [Oppenheim, Appendix], [<https://bit.ly/3aND12K>], [<https://bit.ly/3cSxtCi>].

For system (b), we can derive the Frequency response using similar approach

$$\begin{aligned} \text{(b)} \quad \mathcal{F} \left\{ \frac{dy(t)}{dt} + 2y(t) = x(t) \right\} &\Leftrightarrow (j2\pi f) Y(f) + 2Y(f) = X(f), \\ \Rightarrow H(f) = \frac{Y(f)}{X(f)} = \frac{1}{2 + j2\pi f} &\xrightarrow{\mathcal{F}^{-1}} h(t) = e^{-2t} u(t), \end{aligned}$$

where we used the FT pairs in Table 4.2. The system output for each input signals are derived as

$$\begin{aligned} Y_1(f) &= \frac{1}{(1 + j2\pi f)(2 + j2\pi f)} = \frac{1}{1 + j2\pi f} - \frac{1}{2 + j2\pi f} \Rightarrow y_1(t) = e^{-t} u(t) - e^{-2t} u(t), \\ Y_2(f) &= \frac{1}{(2 + j2\pi f)^3} \Rightarrow y_2(t) = \frac{1}{2} t^2 e^{-2t} u(t), \\ Y_3(f) &= \frac{1}{4} \delta(f) + \frac{1}{j2\pi f(2 + j2\pi f)} = \frac{1}{2} \left(\frac{1}{2} \delta(f) + \frac{1}{j2\pi f} \right) - \frac{1}{2} \left(\frac{1}{2 + j2\pi f} \right) \Rightarrow y_3(t) = \frac{1}{2} u(t) - \frac{1}{2} e^{-2t} u(t). \end{aligned}$$

For system (c), the Frequency response and impulse response are derived as follows

$$\begin{aligned} \text{(c)} \quad \mathcal{F} \left\{ \frac{d^2 y(t)}{dt^2} + 4 \frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t) \right\} &\Leftrightarrow \{(j2\pi f)^2 + 4(j2\pi f) + 3\} Y(f) = \{j2\pi f + 2\} X(f), \\ \Rightarrow H(f) = \frac{Y(f)}{X(f)} = \frac{2 + j2\pi f}{(j2\pi f)^2 + 4(j2\pi f) + 3} &= \frac{1/2}{3 + j2\pi f} + \frac{1/2}{1 + j2\pi f} \xrightarrow{\mathcal{F}^{-1}} h(t) = \frac{1}{2} e^{-3t} u(t) + \frac{1}{2} e^{-t} u(t), \end{aligned}$$

where we used the FT pairs in Table 4.2. The system output for each input signals are derived as

$$\begin{aligned} Y_1(f) &= \frac{2 + j2\pi f}{(1 + j2\pi f)^2 (3 + j2\pi f)} = \frac{1/4}{1 + j2\pi f} + \frac{1/2}{(1 + j2\pi f)^2} + \frac{-1/4}{3 + j2\pi f} \\ \xrightarrow{\mathcal{F}^{-1}} y_1(t) &= \frac{1}{4} e^{-t} u(t) + \frac{1}{2} t e^{-t} u(t) - \frac{1}{4} e^{-3t} u(t), \\ Y_2(f) &= \frac{1}{(1 + j2\pi f)(2 + j2\pi f)(3 + j2\pi f)} = \frac{1/2}{1 + j2\pi f} + \frac{-1}{2 + j2\pi f} + \frac{1/2}{3 + j2\pi f} \\ \xrightarrow{\mathcal{F}^{-1}} y_2(t) &= \frac{1}{2} e^{-t} u(t) - e^{-2t} u(t) + \frac{1}{2} e^{-3t} u(t), \\ Y_3(f) &= \frac{1}{3} \delta(f) + \frac{2 + j2\pi f}{j2\pi f(1 + j2\pi f)(3 + j2\pi f)} = \frac{1}{3} \delta(f) + \frac{2/3}{j2\pi f} + \frac{-1/2}{1 + j2\pi f} + \frac{-1/6}{3 + j2\pi f} \\ &= \frac{2}{3} \left(\frac{1}{2} \delta(f) + \frac{1}{j2\pi f} \right) + \frac{-1/2}{1 + j2\pi f} + \frac{-1/6}{3 + j2\pi f} \\ \xrightarrow{\mathcal{F}^{-1}} y_3(t) &= \frac{2}{3} u(t) - \frac{1}{2} e^{-t} u(t) - \frac{1}{6} e^{-3t} u(t). \end{aligned}$$

For system (d), we can derive the Frequency response using similar approach

$$\begin{aligned} \text{(d)} \quad \mathcal{F} \left\{ \frac{d^2 y(t)}{dt^2} + 6 \frac{dy(t)}{dt} + 8y(t) = 2x(t) \right\} &\Leftrightarrow \{(j2\pi f)^2 + 6(j2\pi f) + 8\} Y(f) = 2X(f), \\ \Rightarrow H(f) = \frac{Y(f)}{X(f)} = \frac{2}{(j2\pi f)^2 + 6(j2\pi f) + 8} &= \frac{2}{(4 + j2\pi f)(2 + j2\pi f)} = \frac{1}{2 + j2\pi f} + \frac{-1}{4 + j2\pi f} \\ \xrightarrow{\mathcal{F}^{-1}} h(t) &= e^{-2t} u(t) - e^{-4t} u(t) \end{aligned}$$

where we used the FT pairs in Table 4.2. The system output for each input signals are derived as

$$\begin{aligned}
 Y_1(f) &= \frac{2}{(1+j2\pi f)(2+j2\pi f)(4+j2\pi f)} = \frac{2/3}{1+j2\pi f} + \frac{-1}{2+j2\pi f} + \frac{1/3}{4+j2\pi f} \\
 \xrightarrow{\mathcal{F}^{-1}} y_1(t) &= \frac{2}{3}e^{-t}u(t) - e^{-2t}u(t) + \frac{1}{3}e^{-4t}u(t), \\
 Y_2(f) &= \frac{2}{(2+j2\pi f)^3(4+j2\pi f)} = \frac{1/4}{2+j2\pi f} + \frac{-1/2}{(2+j2\pi f)^2} + \frac{1}{(2+j2\pi f)^3} + \frac{-1/4}{4+j2\pi f} \\
 \xrightarrow{\mathcal{F}^{-1}} y_2(t) &= \frac{1}{4}e^{-2t}u(t) - \frac{1}{2}te^{-2t}u(t) + \frac{1}{2}t^2e^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t), \\
 Y_3(f) &= \frac{1}{8}\delta(f) + \frac{2}{j2\pi f(2+j2\pi f)(4+j2\pi f)} = \frac{1}{4}\left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right) + \frac{-1/2}{2+j2\pi f} + \frac{1/4}{4+j2\pi f} \\
 \xrightarrow{\mathcal{F}^{-1}} y_3(t) &= \frac{1}{4}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{4}e^{-4t}u(t).
 \end{aligned}$$



4.3 Nyquist Sampling

Given a CT signal $x(t)$, if we measure its values every T time units, then we get the following samples

$$\dots, x(-2T), x(-T), x(0), x(T), x(2T), \dots$$

This process is called [Sampling](#).

$$\text{Sampling period: } T, \quad \text{Sampling frequency: } f_s = \frac{1}{T}$$

In general, some information is lost in this process if the signal is not bandlimited (the bandwidth is not finite).

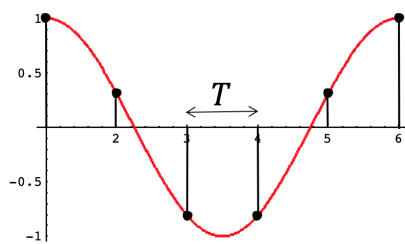
Sampling Theorem

Let $x(t)$ be a band-limited signal with $X(f) = 0$ for $|f| > f_M$. Then $x(t)$ is uniquely determined by its samples $\{x(-kT), \text{arbitrary integer } k\}$ if the following condition is satisfied

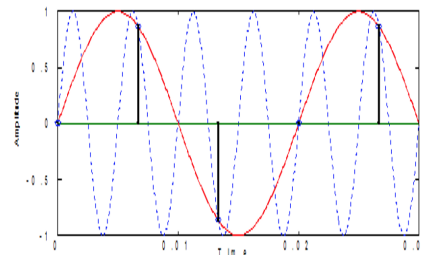
$$f_s \geq 2f_M, \quad (4.12)$$

where $f_M, f_s, 2f_M$ are the signal bandwidth, sampling frequency, [Nyquist rate](#) (minimum f_s), respectively.

✳ **Intuitive Explanation:** Sampling theorem in (4.12) indicates that the sampling rate should be higher than the frequency content of the signal. Otherwise, the recovered signal after sampling will suffer significant distortion.



Sampling rate is high enough.



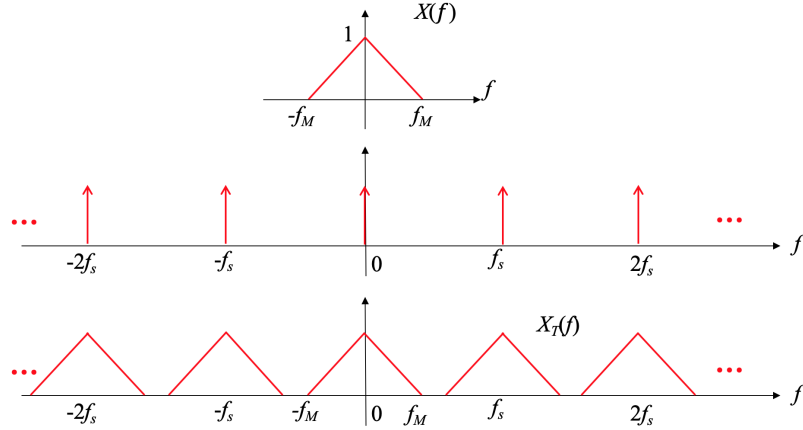
Sampling rate is too low!

*** Formal Explanation:** Analytically, Sampling process is multiplication of the bandlimited signal $x(t)$ and the periodic impulse train $\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$, which we will denote as $x_T(t)$

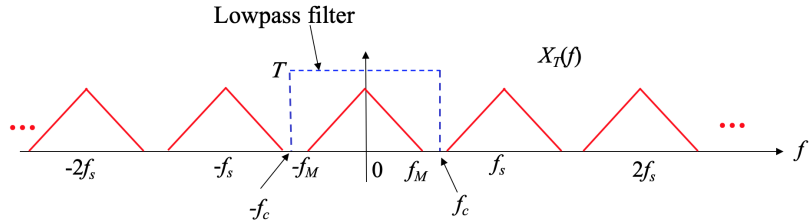
$$x_T(t) = x(t)\delta_T(t) = \sum_{k=-\infty}^{\infty} x(kT) \delta(t - kT).$$

By using the multiplication property, the FT of the sampled signal $x_T(t)$ is obtained as follows

$$X_T(f) = \mathcal{F}(x(t)) * \mathcal{F}(\delta_T(t)) = X(f) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(f - kf_s) = f_s \sum_{k=-\infty}^{\infty} X(f - kf_s), \quad f_s \triangleq \frac{1}{T}$$



The shifted replicas do not overlap if and only if $f_s \geq 2f_M$. If (4.12) is satisfied, then $x(t)$ can be exactly recovered from $x_T(t)$ by using an ideal lowpass filter with amplitude $\frac{1}{f_s}$ and cutoff frequency f_c where $f_M \leq f_c \leq f_s - f_M$. In other words, ideal lowpass filter (or equivalently, sinc interpolation) can exactly reproduce the original signal.



[Example 4-7] Find the Nyquist sampling rate for the following signals

a) $x(t) = 2 \cos(200\pi t) + \sin(500\pi t)$

b) $x(t) = 2 \cos^2(200\pi t)$

Solution) In (a), the signal bandwidth for $\cos(200\pi t)$ and $\sin(500\pi t)$ are respectively given by

$$f_1 = \frac{200\pi}{2\pi} = 100, \quad f_2 = \frac{500\pi}{2\pi} = 250 \Rightarrow f_M = 250$$

and the bandwidth of the $x(t)$ is $f_M = 250$. Hence the Nyquist sampling rate for (a) is $2f_M = 500$. In (b), $2 \cos^2(200\pi t) = 1 + \cos(400\pi t)$ whose signal bandwidth is $f_M = \frac{400\pi}{2\pi} = 200$. Hence the Nyquist sampling rate for (b) is $2f_M = 400$.

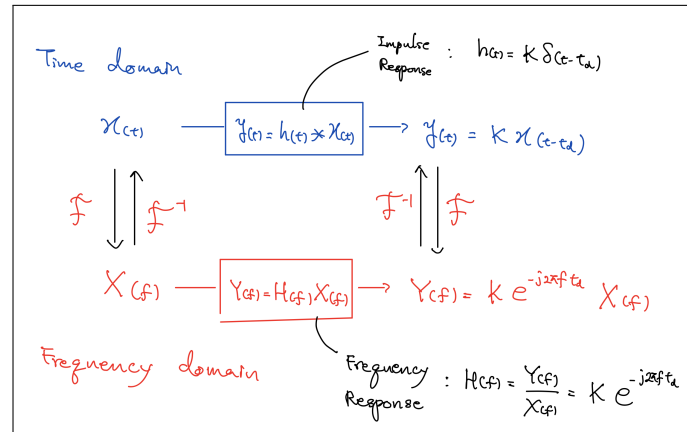


4.4 Filtering

※ **Distortionless Transmission:** In order to have a distortionless transmission, the output of an LTI system needs to have an identical shape to the input signal, although its amplitude may be different and it may be delayed in time.

$$\text{Input signal } x(t) \Rightarrow \text{Output Signal } y(t) = Kx(t - t_d),$$

where t_d is the time delay, K is a gain constant. By using the FT, we can obtain the Frequency response as follows

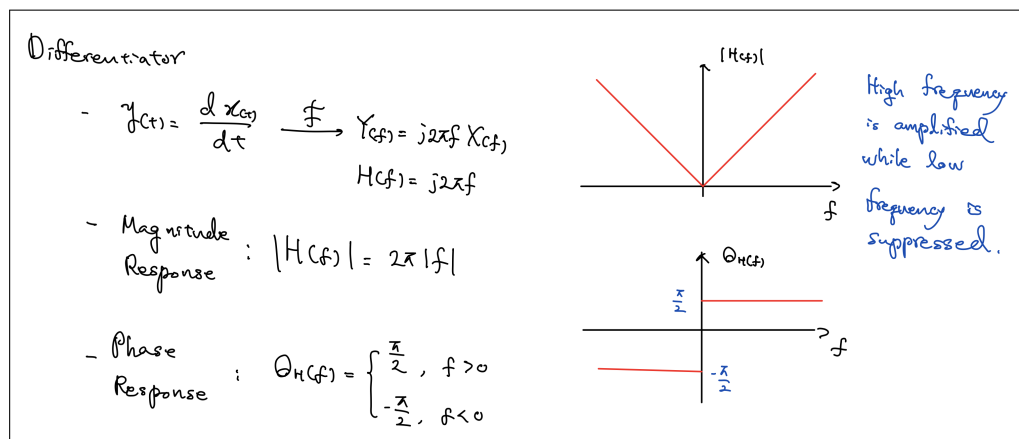


Hence, the systems with distortionless transmission should have Frequency response $H(f) = Ke^{-j2\pi f t_d}$, constant magnitude response $|H(f)| = K$, and linear phase response $\theta_H(f) = -j2\pi f t_d$ over the entire frequency range.

※ **Filtering** is a process that changes the amplitude (or phase) of some frequency components of an input signal.

1. Frequency-Shaping Filter

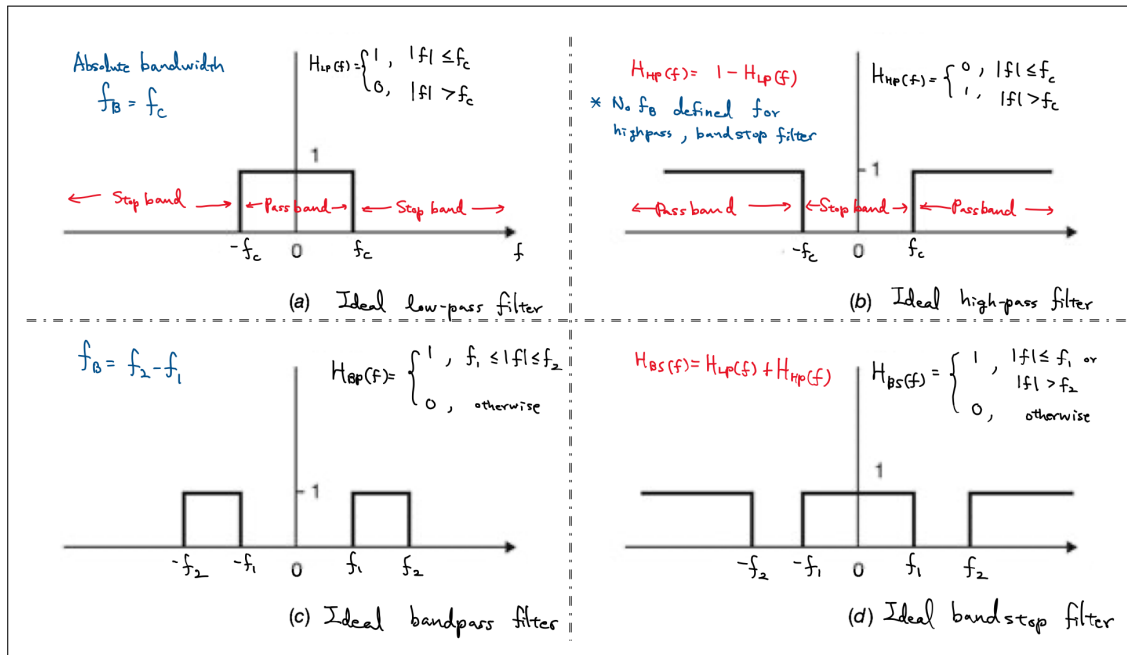
- amplify some frequency components while suppress some other frequency components
- e.g. differentiator, equalizer in a Hi-Fi system



2. Frequency-Selective Filter

- select some bands of frequencies and reject others.
- e.g. low-pass filter, high-pass filter, band-pass filter, band-stop filter

※ **Ideal Frequency Selective Filter:** An ideal frequency selective filter is one that passes signals at one set of frequencies (referred as the *pass band*) and completely rejects the rest (referred as the *stop band*).



a) Ideal Low-Pass Filter

$$H(f) = \begin{cases} 1, & |f| \leq f_c \\ 0, & |f| > f_c \end{cases}$$

where w_c is the *cutoff frequency*.

b) Ideal High-Pass Filter

$$H(f) = \begin{cases} 0, & |f| \leq f_c \\ 1, & |f| > f_c \end{cases}$$

c) Ideal Bandpass Filter

$$H(f) = \begin{cases} 1, & f_1 < |f| < f_2 \\ 0, & \text{otherwise} \end{cases}$$

d) Ideal Bandstop Filter

$$H(f) = \begin{cases} 0, & f_1 < |f| < f_2 \\ 1, & \text{otherwise} \end{cases}$$

* **Bandwidth:** There are many different definitions of filter bandwidth.

1. Absolute Bandwidth f_B

- absolute BW (Bandwidth) of an ideal low-pass filter: $f_B = f_c$
- absolute BW of an ideal bandpass filter: $f_B = f_2 - f_1$
- absolute BS of an ideal high-pass or bandstop filter are not defined.

2. 3-dB Bandwidth (*Half-Power Bandwidth*) $f_3 \text{ dB}$

- $f_3 \text{ dB}$ is the frequency where the peak magnitude spectrum $|H(0)|$ drops to $|H(0)|/\sqrt{2}$. Since the power is proportional to the square of $|H(f)|$, i.e., $P \propto |H(f)|^2$, then the 3-dB bandwidth represent the frequency where the peak power reduces to the half.

$$\begin{aligned}
 10 \log_{10} \left(\frac{P(f_3 \text{ dB})}{P(f_{\max})} \right) \text{ dB} &= 10 \log_{10} \left(\frac{|H(f_3 \text{ dB})|^2}{|H(0)|^2} \right) \text{ dB} \xrightarrow{|H(f_3 \text{ dB})| = \frac{|H(0)|}{\sqrt{2}}} 10 \log_{10} (1/2) \text{ dB} \\
 &= -10 \log_{10} (2) \text{ dB} \approx -3 \text{ dB}
 \end{aligned}$$

3. Equivalent Bandwidth f_{eq} . Refer [Schaum's text, Problem 5.55]

- f_{eq} is defined as the bandwidth of an ideal filter where the power of the ideal filter is equal to that of the real filter given the same input signals as follows

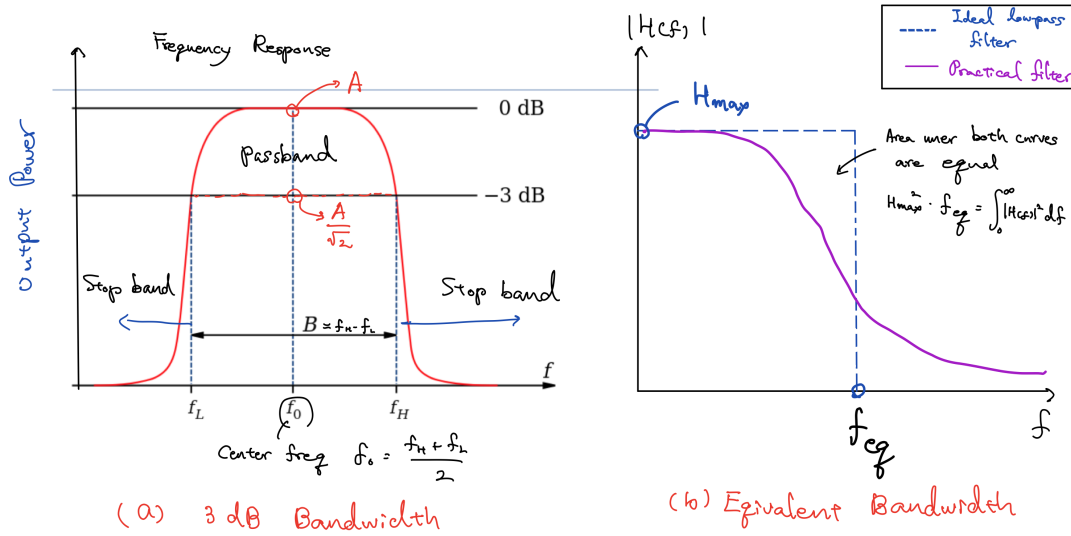
$$f_{eq} = \int_0^\infty \frac{|H(f)|^2}{|H_{max}|^2} df$$

where H_{max} denotes the maximum value of the magnitude spectrum.

4. x percent Energy containment Bandwidth $f_{x\%}$. (e.g., 90 percent energy containment BW $f_{90\%}$)

$$\int_{-f_{x\%}}^{f_{x\%}} |X(f)|^2 df = (x\%) E_x \quad \text{where} \quad E_x = \int_{-\infty}^{\infty} |X(f)|^2 df.$$

Refer [Schaum's text, Problem 5.57]



[Example 4-8] Consider the following filters with impulse response $h(t)$. For each filter, derive the 3-dB bandwidth $f_{3\text{ dB}}$, equivalent bandwidth f_{eq} , and 90 percent energy containment bandwidth $f_{90\%}$, respectively.

a) $h(t) = w_0 e^{-w_0 t} u(t)$ where $w_0 = 2\pi f_0$

b) $h(t) = \frac{1}{\alpha^2 + (2\pi t)^2}$ where $\alpha > 0$

Solution) For (a), the frequency response $H(f)$ is obtained as

$$(a) h(t) = w_0 e^{-w_0 t} u(t) \Leftrightarrow H(f) = \frac{w_0}{w_0 + j2\pi f} = \frac{1}{1 + j(f/f_0)} \quad \text{where } f_0 = \frac{w_0}{2\pi}$$

Since the maximum magnitude spectrum is $|H(0)| = 1$, the 3-dB bandwidth $f_{3\text{ dB}}$, the equivalent bandwidth f_{eq} , and the 90 percent energy containment bandwidth $f_{90\%}$ occurs when the following conditions are satisfied.

$$\left\{ \begin{array}{l} f_{3\text{ dB}}; \quad |H(f_{3\text{ dB}})| = \frac{|H(0)|}{\sqrt{2}}, \\ f_{eq}; \quad f_{eq} = \int_0^\infty \frac{|H(f)|^2}{|H(0)|^2} df, \\ f_{90\%}; \quad \int_{-f_{90\%}}^{f_{90\%}} |H(f)|^2 df = 0.9 \int_{-\infty}^{\infty} |H(f)|^2 df \end{array} \right. \quad , \quad \text{where } |H(f)| = \frac{1}{\sqrt{1 + (f/f_0)^2}} \quad (4.13)$$

Since $|H(f_{3\text{ dB}})| = \frac{1}{\sqrt{1+(f_{3\text{ dB}}/f_0)^2}} = \frac{1}{\sqrt{2}}$ when $f_{3\text{ dB}} = f_0$, the 3-dB bandwidth is equal to f_0 . By using (4.14), we obtain

$$f_{\text{eq}} = \int_0^\infty \frac{1}{1+(f/f_0)^2} df = f_0 \int_0^\infty \frac{1}{1+x^2} dx = \frac{f_0 \pi}{2},$$

$$\text{LHS of } f_{90\%} \text{ in (1.13)} \int_{-f_{90\%}}^{f_{90\%}} |H(f)|^2 df = 2 \int_0^{f_{90\%}} \frac{1}{1+(f/f_0)^2} df = 2f_0 \tan^{-1}(x) \Big|_0^{f_{90\%}/f_0} = 2f_0 \tan^{-1}\left(\frac{f_{90\%}}{f_0}\right),$$

$$\text{RHS of } f_{90\%} \text{ in (1.13)} 0.9 \int_{-\infty}^\infty \frac{1}{1+(f/f_0)^2} df = 0.9 * 2 \int_0^\infty \frac{1}{1+(f/f_0)^2} df = 0.9 f_0 \pi \Rightarrow f_{90\%} = f_0 \tan(0.45\pi)$$

Hence, the three BWs $f_{3\text{ dB}}$, f_{eq} , and $f_{90\%}$ are given by

$$(a) f_{3\text{ dB}} = f_0, \quad f_{\text{eq}} = \frac{f_0 \pi}{2} \simeq 1.57 f_0, \quad f_{90\%} = f_0 \tan(0.45\pi) \simeq 6.31 f_0, \quad \Leftrightarrow \quad f_{3\text{ dB}} < f_{\text{eq}} < f_{90\%}$$

Similarly, for (b), the frequency response $H(f)$ is obtained as

$$(b) h(t) = \frac{1}{\alpha^2 + (2\pi t)^2} \quad \Leftrightarrow \quad H(f) = \frac{1}{2\alpha} e^{-\alpha|f|}$$

and the maximum magnitude spectrum is $|H(0)| = \frac{1}{2\alpha}$. Then, based on (4.13), we obtain

$$|H(f_{3\text{ dB}})| = \frac{|H(0)|}{\sqrt{2}} \Rightarrow \frac{1}{2\alpha} e^{-\alpha|f_{3\text{ dB}}|} = \frac{1}{2\alpha\sqrt{2}} \Rightarrow f_{3\text{ dB}} = \frac{1}{\alpha} \ln \sqrt{2},$$

$$f_{\text{eq}} = \int_0^\infty e^{-2\alpha f} df = \frac{1}{2\alpha},$$

$$\text{LHS of } f_{90\%} \text{ in (1.13)} \int_{-f_{90\%}}^{f_{90\%}} |H(f)|^2 df = 2 \int_0^{f_{90\%}} e^{-2\alpha f} df = \frac{1}{\alpha} [1 - e^{-2\alpha f_{90\%}}],$$

$$\text{RHS of } f_{90\%} \text{ in (1.13)} 0.9 \int_{-\infty}^\infty e^{-2\alpha f} df = 0.9 * 2 \int_0^\infty e^{-2\alpha f} df = \frac{0.9}{\alpha} \Rightarrow f_{90\%} = \frac{\ln(10)}{2\alpha}$$

Hence, the BWs $f_{3\text{ dB}}$, f_{eq} , and $f_{90\%}$ are given by

$$(b) f_{3\text{ dB}} = \frac{1}{\alpha} \ln \sqrt{2} \simeq \frac{0.35}{\alpha}, \quad f_{\text{eq}} = \frac{1}{2\alpha} = \frac{0.5}{\alpha}, \quad f_{90\%} = \frac{\ln(10)}{2\alpha} \simeq \frac{1.15}{\alpha}, \quad \Leftrightarrow \quad f_{3\text{ dB}} < f_{\text{eq}} < f_{90\%}$$

Useful Equality

Refer Wolfram Alpha [<https://bit.ly/3d3RZoA>]

$$\int \frac{1}{a^2 + x^2} dx = \frac{\tan^{-1}\left(\frac{x}{a}\right)}{a}, \quad \int_0^\infty \frac{1}{a^2 + x^2} dx = \frac{\pi}{2|a|} \quad \text{for real-valued } a \quad (4.14)$$

