

Question 1

A vector $\vec{a}$ has magnitude 15.3 and is in the same direction as the line, $\overrightarrow{BC}$, where B has coordinates $(1, 2, -4)$ and C has coordinates $(-1, 0, 3)$. Find the component form of $\vec{a}$.

Question 2

A vector $\vec{a}$ is parallel to the line through $A(1, 2, 3)$ and $B(-1, 0, 2)$, whilst a vector $\vec{b}$ is parallel to the line through $C(0, 0, 1)$ and $D(-1, -2, 3)$. Both vectors have magnitude 4. Find $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ in component form.

Question 3

If the vertices of a triangle are the points $(1, 0, -1)$, $(-1.3, 0, 2)$ and $(2.4, 1, 2)$, what are the vectors that represent the sides? Determine the length of each side.

Question 4

Find the value of μ such that $\vec{x} = 4\hat{i} - 3\hat{j} - 5\hat{k}$, $\vec{y} = \mu\hat{i} - 9\hat{j} - \hat{k}$ and $|\vec{x} - \vec{y}| = \sqrt{61}$.

Question 5

Find a unit vector parallel to the resultant of the vectors $\vec{x} = \hat{i} + \hat{j} + \hat{k}$, $\vec{y} = \hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{z} = \hat{i} - \hat{j} + 2\hat{k}$.

Question 6

A set of rectangular cartesian axes is so arranged that the x-axis points east, the y-axis points north and the z-axis points vertically upwards. Evaluate the scalar products of the vectors $\vec{a}$ and $\vec{b}$ in the following cases.

- (i) $\vec{a}$ is of magnitude 3 and points SE, $\vec{b}$ is of magnitude 2 and points E.
- (ii) $\vec{a}$ is of unit magnitude and points NE, $\vec{b}$ is of magnitude 2 and points vertically upwards.
- (iii) $\vec{a}$ is of unit magnitude and points NE, $\vec{b}$ is of magnitude 2 and points W.

Question 7

Find $a \cdot b$ for the following cases.

- (i) $a(-1, 0, 7)$ and $b(2, 4, -6)$
- (ii) $a = 4\hat{i} - 3\hat{j} + \hat{k}$ and $b = \hat{i} + \hat{j} + \hat{k}$

Question 8

Axes Oxyz are positioned so that Ox points east, Oy points north and Oz points vertically upwards. Find the vector product $\vec{a} \times \vec{b}$ of vectors $\vec{a}$ and $\vec{b}$ in the following cases.

- (i) $\vec{a}$ is of unit magnitude and points E, $\vec{b}$ is of magnitude 2 and points 30° N of E.
- (ii) $\vec{a}$ is of unit magnitude and points E, $\vec{b}$ is of magnitude 2 and points SW.
- (iii) $\vec{a}$ is of unit magnitude and points vertically upwards, $\vec{b}$ is of unit magnitude and points NE.

Question 9

Find $a \times b$ of $a(1, 3, -2)$ and $b(-1, 5, 7)$.

Question 10

Find eigenvalues and the corresponding eigenvectors for the following cases.

(i) $\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(iv) $\begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix}$

(v) $\begin{pmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{pmatrix}$

(vi) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

Solution

Question 1

A vector $\vec{a}$ has magnitude 15.3 and is in the same direction as the line, $\overrightarrow{BC}$, where B has coordinates $(1, 2, -4)$ and C has coordinates $(-1, 0, 3)$. Find the component form of $\vec{a}$.

$$\begin{aligned}\vec{a} &= |\vec{a}|\hat{a} = 15.3\hat{a} = 15.3\left(\frac{1}{\sqrt{57}}\right)(-2\hat{i} - 2\hat{j} + 7\hat{k}) \\ &= -4.053\hat{i} - 4.053\hat{j} + \hat{k}\end{aligned}$$

Question 2

A vector $\vec{a}$ is parallel to the line through $A(1, 2, 3)$ and $B(-1, 0, 2)$, whilst a vector $\vec{b}$ is parallel to the line through $C(0, 0, 1)$ and $D(-1, -2, 3)$. Both vectors have magnitude 4. Find $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ in component form.

$$\begin{aligned}\vec{a} + \vec{b} &= -4\hat{i} - \frac{16}{3}\hat{j} - 4\hat{k} \\ \vec{a} - \vec{b} &= -\frac{4}{3}\hat{i} + \frac{4}{3}\hat{k}\end{aligned}$$

Question 3

If the vertices of a triangle are the points $(1, 0, -1)$, $(-1.3, 0, 2)$ and $(2.4, 1, 2)$, what are the vectors that represent the sides? Determine the length of each side.

$$|\overrightarrow{AB}| = \sqrt{(-2.3)^2 + (3)^2} = 3.78$$

$$|\overrightarrow{AC}| = \sqrt{(1.4)^2 + (3)^2} = 3.31$$

$$|\overrightarrow{BC}| = \sqrt{(3.7)^2 + (1)^2} = 3.83$$

Question 4

Find the value of μ such that $\vec{x} = 4\hat{i} - 3\hat{j} - 5\hat{k}$, $\vec{y} = \mu\hat{i} - 9\hat{j} - \hat{k}$ and $|\vec{x} - \vec{y}| = \sqrt{61}$.

$$\mu = 1 \text{ or } 7.$$

Question 5

Find a unit vector parallel to the resultant of the vectors $\vec{x} = \hat{i} + \hat{j} + \hat{k}$, $\vec{y} = \hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{z} = \hat{i} - \hat{j} + 2\hat{k}$.

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{43}}(3\hat{i} + 3\hat{j} + 5\hat{k})$$

Question 6

A set of rectangular cartesian axes is so arranged that the x-axis points east, the y-axis points north and the z-axis points vertically upwards. Evaluate the scalar products of the vectors $\vec{a}$ and $\vec{b}$ in the following cases.

- (i) $\vec{a}$ is of magnitude 3 and points SE, $\vec{b}$ is of magnitude 2 and points E.
- (ii) $\vec{a}$ is of unit magnitude and points NE, $\vec{b}$ is of magnitude 2 and points vertically upwards.
- (iii) $\vec{a}$ is of unit magnitude and points NE, $\vec{b}$ is of magnitude 2 and points W.

(i)

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta = |3||2| \cos 45^\circ = 3\sqrt{2}$$

(ii)

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta = |1||2| \cos 90^\circ = 0$$

(iii)

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta = |1||2| \cos 135^\circ = -\sqrt{2}$$

Question 7

Compute $a \cdot b$,

(i) $a = (-1, 0, 7)$ & $b = (2, 4, -6)$

(ii) $a = 4i - 3j + k$ & $b = i + j + k$

(i)

$$\vec{a} \cdot \vec{b} = 44$$

(ii)

$$\vec{a} \cdot \vec{b} = 2$$

Question 8

Axes Oxyz are positioned so that Ox points east, Oy points north and Oz points vertically upwards. Find the vector product $\vec{a} \times \vec{b}$ of vectors $\vec{a}$ and $\vec{b}$ in the following cases.

(i) $\vec{a}$ is of unit magnitude and points E, $\vec{b}$ is of magnitude 2 and points 30° N of E.

(ii) $\vec{a}$ is of unit magnitude and points E, $\vec{b}$ is of magnitude 2 and points SW.

(iii) $\vec{a}$ is of unit magnitude and points vertically upwards, $\vec{b}$ is of unit magnitude and points NE.

(i)

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin \theta = |1||2| \sin 30^\circ = 1$$

It is a vertically upwards unit vector.

(ii)

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin \theta = |1||2| \sin 225^\circ = \sqrt{2}$$

It is a vector of magnitude $\sqrt{2}$ pointing downwards.

(iii)

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin \theta = |1||1| \sin 90^\circ = 1$$

It is a unit vector pointing NW.

Question 9

Compute $a \times b$ of $(1, 3, -2), (-1, 5, 7)$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 5 & 7 \end{vmatrix}$$

$$= 31\hat{i} - 5\hat{j} + 8\hat{k}$$

Question 10

Find eigenvalues and the corresponding eigenvectors for the following cases.

(i) $\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(iv) $\begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix}$

(v) $\begin{pmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{pmatrix}$

(vi) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

(i)

Eigenvalues are 1 & -1 .

When $\lambda = 1$,

The corresponding eigenvector of A for $\lambda = 1$ is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}, k \neq 0$.

When $\lambda = -1$,

The corresponding eigenvector of A for $\lambda = -1$ is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \end{pmatrix}, k \neq 0$.

(ii)

Eigenvalues are 0 & -2.

When $\lambda = 0$,

The corresponding eigenvector of A for $\lambda = 0$ is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix}, k \neq 0$.

When $\lambda = -2$,

The corresponding eigenvector of A for $\lambda = -2$ is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}, k \neq 0$.

(iii)

Eigenvalues are 2 & -2 .

When $\lambda = \sqrt{2}$,

The corresponding eigenvector of A for $\lambda = \sqrt{2}$ is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix}, k \neq 0$.

When $\lambda = -\sqrt{2}$,

The corresponding eigenvector of A for $\lambda = -2$ is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix}, k \neq 0$.

(iv)

Eigenvalues are -1 & 2 .

When $\lambda = -1$,

The corresponding eigenvector of A for $\lambda = -1$ is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} -\frac{1}{4} \\ -\frac{3}{4} \\ 1 \end{pmatrix}, k \neq 0$.

When $\lambda = 2$,

The corresponding eigenvector of A for $\lambda = 2$ is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, k \neq 0$.

(v)

Eigenvalues are 1 , 6 & 7 .

When $\lambda = 1$,

The corresponding eigenvector of A for $\lambda = 1$ is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} \frac{15}{16} \\ -\frac{1}{2} \\ 1 \end{pmatrix}, k \neq 0$.

When $\lambda = 6$,

The corresponding eigenvector of A for $\lambda = 6$ is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, k \neq 0$.

When $\lambda = 7$,

The corresponding eigenvector of A for $\lambda = 7$ is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, k \neq 0$.

(vi)

Eigenvalues are 1 & 4.

When $\lambda = 1$,

The corresponding eigenvectors of A for

$$\lambda = 1 \text{ are } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, k \neq 0.$$

When $\lambda = 4$,

$$\text{The corresponding eigenvector of } A \text{ for } \lambda = 4 \text{ is } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, k \neq 0.$$