

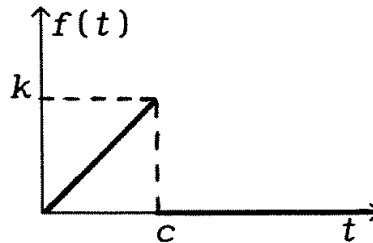
Problem Set 3: Laplace Transform

1. Directly from definition, find the Laplace transform of

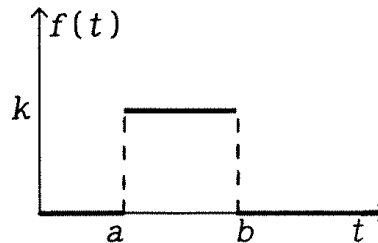
(a) $\sin(\omega t + \theta)$

(b) $\cosh^2 3t$

(c)



(d)



(e) Expand $\sin(\omega t + \theta)$ and use the linearity property of Laplace transform to check the result of (a).

2. Find the Laplace transforms of

(a) $t^6 - 3t^4 + t^2$

(b) $\sin^2 t$

(c) $\sin t \cos t$

(d) $e^t \sin^2 t$

(e) $e^{-\alpha t} (A \cos \omega t + B \sin \omega t)$

3. Find $f(t)$ if $\mathcal{L}[f] = F(s)$ is:

(a) $\frac{s-4}{s^2-4}$

(b) $\frac{1}{(s-a)(s-b)}$ where $a \neq b$.

4. Use the result for the Laplace transforms of derivatives to find

(a) $\mathcal{L}[t \cos \omega t]$

(b) the solution of
$$\begin{cases} y'' - 3y' + 2y = 4t \\ y(0) = 4, y'(0) = 3 \end{cases}$$

5. Find the inverse Laplace transform of

(a) $\frac{2}{(s-2)^2}$

(b) $\frac{3}{2s^2-8s+10}$

(c) $\frac{s+6}{s^2+4s+12}$

(d) $\frac{1+e^{-2s}}{s^2}$

6. (a) Find $\sin t * \cos t$.

(b) Using Convolution Theorem, find $\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+4)^2}\right]$.

7. Solve the following initial value problem by Laplace transform:

$$\begin{cases} y'' - 2y' + y = e^x \\ y(0) = -2, y'(0) = -3 \end{cases}$$

8. Solve the system $\begin{cases} x'' - 2x' + 3y' + 2y = 4 \\ 2y' - x' + 3y = 0 \\ x(0) = x'(0) = y(0) = 0 \end{cases}$ by applying the Laplace transform to the differential equations, incorporating the initial conditions.

9. Consider the initial-value problem $\begin{cases} \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = \alpha(x) \dots\dots(I) \\ y(-1) = y'(-1) = 0 \end{cases}$, where

$$\alpha(x) = \begin{cases} 1 & \text{for } -1 \leq x < 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) (i) Convert the initial-value problem (I) into an initial-value problem

$$\begin{cases} \frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = \beta(t) \dots\dots(II). \\ y(0) = y'(0) = 0 \end{cases}$$

(ii) What is $\beta(t)$?

Consider the initial-value problem (II): $\begin{cases} \frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = \beta(t) \dots\dots(II), \\ y(0) = y'(0) = 0 \end{cases}$, let $B(s)$ and $Y(s)$ be

the Laplace transformations of $\beta(t)$ and $y(t)$, respectively.

(b) Find $Y(s)$ in terms of $B(s)$.

(c) Using partial fractions, express $Y(s)$ in the form $Y(s) = \frac{a}{s-\lambda}B(s) + \frac{b}{s-\mu}B(s)$.

(d) Using the convolution theorem and the expression in (c), find the particular solution $y(t)$

$$\text{to the initial-value problem } \begin{cases} \frac{d^2 y}{dt^2} + \frac{dy}{dt} - 2y = \beta(t) \dots\dots (II) \\ y(0) = y'(0) = 0 \end{cases}$$

(e) Find the particular solution to the initial-value problem $\begin{cases} \frac{d^2 y}{dx^2} + \frac{dy}{dx} - 2y = \alpha(x) \dots\dots (I) \\ y(-1) = y'(-1) = 0 \end{cases}$

10. Solve the integral equation $f(t) = at + \int_0^t \sin(t-\tau) f(\tau) d\tau$.

11. Solve $\begin{cases} y' - 4y + 4 \int_0^t y dt = \frac{1}{3} t^3 \\ y(0) = 0 \end{cases}$

12. Let $f(t) = \int_0^t u e^{-2u} \cos 3u du$. Show by computation that $\mathcal{L}[f(t)] = \frac{(s+2)^2 - 9}{s[(s+2)^2 + 9]^2}$.

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