

1. (a)

$$\begin{aligned}
 xy \frac{dy}{dx} &= 1 + x^2 + y^2 + x^2 y^2 \Rightarrow \int \frac{y dy}{1 + y^2} = \int \left(\frac{1}{x} + x \right) dx \\
 &\Rightarrow \frac{1}{2} \int \frac{d(1 + y^2)}{1 + y^2} = \int \left(\frac{1}{x} + x \right) dx \Rightarrow \frac{1}{2} \ln(1 + y^2) = \ln|x| + \frac{x^2}{2} + c \\
 &\Rightarrow 1 + y^2 = c_1 x^2 e^{x^2}, \quad c_1 = e^{2c}
 \end{aligned}$$

(b) $\frac{dy}{dx} = \frac{y-x}{y+x}$. Let $y = xv$. Then $\frac{dy}{dx} = x \frac{dv}{dx} + v \Rightarrow x \frac{dv}{dx} + v = \frac{v-1}{v+1} \Rightarrow -\int \frac{v+1}{v^2+1} dv = \int \frac{dx}{x}$

$$\begin{aligned}
 &-\left(\int \frac{\frac{1}{2} d(v^2+1)}{v^2+1} + \int \frac{dv}{v^2+1} \right) = \int \frac{dx}{x} \Rightarrow -\left[\frac{1}{2} \ln(v^2+1) + \tan^{-1} v \right] = \ln|x| + c_1 \\
 &\Rightarrow -\left[\frac{1}{2} \ln\left(\frac{y^2}{x^2} + 1\right) + \tan^{-1} \frac{y}{x} \right] = \ln|x| + c_1 \\
 &\text{Or } \frac{1}{2} \ln(y^2 + x^2) + \tan^{-1} \frac{y}{x} = c, \text{ where } c = -c_1
 \end{aligned}$$

(c) $\cos x \frac{dy}{dx} + y = \sin^2 x \Rightarrow \frac{dy}{dx} + (\sec x) y = \sin^2 x \sec x$

An integrating factor is $I(x) = e^{\int \sec x dx} = e^{\ln(\sec x + \tan x)} = \sec x + \tan x$.

$$\begin{aligned}
 (\sec x + \tan x) y &= \int (\sec x + \tan x) \sin^2 x \sec x dx = \int \left(\tan^2 x + \frac{\sin^3 x}{\cos^2 x} \right) dx \\
 &= \int (\sec^2 x - 1) dx + \int \frac{(1 - \cos^2 x) \sin x}{\cos^2 x} dx \\
 &= \int \sec^2 x dx - \int 1 dx + \int \frac{\sin x}{\cos^2 x} dx - \int \sin x dx \\
 &= \tan x - x - \left(\int (\cos x)^{-2} d(\cos x) \right) + \cos x \\
 &= \tan x - x + \frac{1}{\cos x} + \cos x + c
 \end{aligned}$$

Therefore, the general solution of the differential equation is

$$y = 1 + \frac{c - x + \cos x}{\sec x + \tan x}$$

(d) $2 \frac{dy}{dx} - y \cot = y^3 \cos x \Rightarrow 2y^{-3} \frac{dy}{dx} - y^{-2} \cot = \cos x$.

Using the transformation $y^{-2} = v$, $-2y^{-3} \frac{dy}{dx} = \frac{dv}{dx}$.

Then, $2y^{-3} \frac{dy}{dx} - y^{-2} \cot = \cos x \Rightarrow \frac{dv}{dx} + v \cot = -\cos x$.

An integrating factor is

$$I(x) = e^{\int \cot x \, dx} = e^{\int \frac{\cos x}{\sin x} \, dx} = e^{\int \frac{1}{\sin x} d(\sin x)} = e^{\ln(\sin x)} = \sin x.$$

Then

$$\begin{aligned} (\sin x) v &= \int -\sin x \cos x \, dx = -\int \sin x \, d(\sin x) = -\frac{\sin^2 x}{2} + c \\ \Rightarrow v &= -\frac{\sin x}{2} + \frac{c}{\sin x} \end{aligned}$$

Therefore, the general solution is $y^{-2} = -\frac{\sin x}{2} + \frac{c}{\sin x}$.

(e) Rewrite the differential equation as

$$(x+y)^{-1} + [(x+y)^{-1} + \tan y] \frac{dy}{dx} = 0$$

Let $M(x, y) = (x+y)^{-1}$ and $N(x, y) = (x+y)^{-1} + \tan y$.

Since $\frac{\partial M}{\partial y} = -(x+y)^{-2} = \frac{\partial N}{\partial x}$, the differential equation is exact.

Then there exists $u(x, y)$ such that

$$\begin{cases} \frac{\partial u}{\partial x} = M(x, y) = (x+y)^{-1} & \dots (1) \\ \frac{\partial u}{\partial y} = N(x, y) = (x+y)^{-1} + \tan y & \dots (2) \end{cases}$$

Integrating (1) w.r.t. x :

$$u(x, y) = \int (x+y)^{-1} \, dx = \ln|x+y| + f(y)$$

Partially differentiate w.r.t. y :

$$\frac{\partial u}{\partial y} = (x+y)^{-1} + f'(y)$$

Comparing with (2):

$$(x+y)^{-1} + f'(y) = (x+y)^{-1} + \tan y$$

$$\Rightarrow f'(y) = \tan y$$

$$\Rightarrow f(y) = \int \tan y \, dy = \int \frac{\sin y}{\cos y} \, dy = -\int \frac{1}{\cos y} d(\cos y) = -\ln|\cos y|$$

$$\therefore u(x, y) = \ln|x+y| - \ln|\cos y|.$$

Therefore, the general solution is

$$\begin{aligned} \ln|x+y| - \ln|\cos y| &= c \Rightarrow \ln \left| \frac{x+y}{\cos y} \right| = c \\ &\Rightarrow \frac{x+y}{\cos y} = k \end{aligned}$$

where k is an arbitrary constant.

(f) $\frac{dy}{dx} = \frac{2x-y-4}{y-2x-2} = \frac{2x-y-4}{-2x+y-2}$

Note that $\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} = 2 - 2 = 0$.

Let $v = 2x - y$. Then $\frac{dv}{dx} = 2 - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 2 - \frac{dv}{dx}$.

The original differential equation becomes

$$\begin{aligned} 2 - \frac{dv}{dx} &= \frac{v-4}{-v-2} \Rightarrow \frac{dv}{dx} = 2 - \frac{v-4}{-v-2} = 2 + \frac{v-4}{v+2} = \frac{3v}{v+2} \\ &\Rightarrow \int \frac{v+2}{v} \, dv = \int 3 \, dx \\ &\Rightarrow v + 2 \ln|v| = 3x + C \end{aligned}$$

Therefore, the general solution is

$$2x - y + 2 \ln|2x - y| = 3x + C$$

$$\Rightarrow -(x + y) + \ln[(2x - y)^2] = C$$

(g) $\frac{dy}{dx} = \frac{x-2y}{2x-y-3}$

Note that $\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} = -1 - (-4) = 3 \neq 0$.

Let $x = X + h$, $y = Y + k$.

Choose h, k so that $\begin{cases} h - 2k = 0 \\ 2h - k - 3 = 0 \end{cases} \Rightarrow \dots \Rightarrow h = 2, k = 1$

The original differential equation becomes

$$\frac{dY}{dX} = \frac{X - 2Y}{2X - Y} = \frac{1 - 2\left(\frac{Y}{X}\right)}{2 - \left(\frac{Y}{X}\right)} \leftarrow \text{Homogeneous equation of degree 0}$$

Let $Y = vX$, then $\frac{dY}{dX} = v + X \frac{dv}{dX}$. The differential equation becomes

$$v + X \frac{dv}{dX} = \frac{1-2v}{2-v} \Rightarrow \dots \Rightarrow \int \frac{2-v}{v^2-4v+1} dv = \int \frac{1}{X} dX$$

$$\Rightarrow \dots \Rightarrow -\frac{1}{2} \ln|v^2 - 4v + 1| = \ln|X| + \ln C_1$$

$$\Rightarrow \dots \Rightarrow (v^2 - 4v + 1)X^2 = C_2, \text{ where } C_2 = \frac{1}{C_1^2}.$$

Substitute $v = \frac{y-k}{x-h} = \frac{y-1}{x-2}$ and $X = x-2$,

the general solution is $\left[\left(\frac{y-1}{x-2} \right)^2 - 4 \left(\frac{y-1}{x-2} \right) + 1 \right] (x-2)^2 = C_2$

or $(y-1)^2 - 4(y-1)(x-2) + (x-2)^2 = C_2$

2. (a) $(x^2 + 1) \frac{dy}{dx} + 4xy = x$ with $y(2) = 1$ $\leftarrow$ Separable equation / First order linear ODE

$$\Rightarrow \dots \Rightarrow \int \frac{1}{4y-1} dy = \int \frac{-x}{x^2+1} dx$$

$$\Rightarrow \dots \Rightarrow \frac{1}{4} \ln|4y-1| = -\frac{1}{2} \ln(x^2+1) + \ln C_1$$

$$\Rightarrow \dots \Rightarrow 4y-1 = \frac{C_1^4}{(x^2+1)^2}$$

The general solution is $y = \frac{1}{4} + \frac{C}{(x^2+1)^2}$, where $C = \frac{C_1^4}{4}$.

With initial condition $y(2) = 1 \Rightarrow \dots \Rightarrow C = \frac{75}{4}$

The particular solution is $y = \frac{1}{4} + \frac{75}{4(x^2+1)^2}$

(b) $(x-2)\frac{dy}{dx} - y = (x-2)^3$ with $y(4)=10$ ← First order linear ODE

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x-2} = (x-2)^2$$

Integrating factor: $I(x) = e^{\int \frac{-1}{x-2} dx} = \dots = \frac{1}{x-2}$

Then we have $\frac{d}{dx} \left[\frac{y}{x-2} \right] = x-2$

$$\Rightarrow \frac{y}{x-2} = \int (x-2) dx = \dots = \frac{(x-2)^2}{2} + C$$

The general solution is $y = \frac{(x-2)^3}{2} + C(x-2)$

With initial condition $y(4)=10 \Rightarrow \dots \Rightarrow C=3$.

The particular solution is $y = \frac{(x-2)^3}{2} + 3(x-2)$

(c) $xy \frac{dy}{dx} = 2(x^2 + y^2)$ with $y(2)=-1$

$$\Rightarrow \frac{dy}{dx} = \frac{2(x^2 + y^2)}{xy} = \frac{2 \left[1 + \left(\frac{y}{x} \right)^2 \right]}{\left(\frac{y}{x} \right)} \quad \leftarrow \text{Homogeneous equation of degree 0}$$

Let $y = vx$, then $\frac{dy}{dx} = v + x \frac{dv}{dx}$.

The original differential equation becomes

$$v + x \frac{dv}{dx} = \frac{2(1+v^2)}{v} \quad \leftarrow \text{Separable equation}$$

$$\Rightarrow \dots \Rightarrow \int \frac{v}{2+v^2} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \dots \Rightarrow \frac{1}{2} \ln(2+v^2) = \ln|x| + \ln C_1$$

$$\Rightarrow \dots \Rightarrow 2+v^2 = Cx^2, \text{ where } C = C_1^2$$

The general solution is $2 + \left(\frac{y}{x} \right)^2 = Cx^2 \Rightarrow y^2 = Cx^4 - 2x^2$

With initial condition $y(2)=-1 \Rightarrow C = \frac{9}{16}$

The particular solution is $y^2 = \frac{9}{16}x^4 - 2x^2$

3. If the differential equation is exact, we have

$$\frac{\partial}{\partial y}(Cx^2ye^y + 2\cos y) = \frac{\partial}{\partial x}(x^3e^yy + x^3e^y + Kx\sin y)$$

$$\Rightarrow Cx^2e^y + Cx^2ye^y - 2\sin y = 3x^2e^yy + 3x^2e^y + K\sin y \Rightarrow \begin{cases} K = -2 \\ C = 3 \end{cases}$$

Our equation is:

$$(3x^2ye^y + 2\cos y)dx + (x^3e^yy + x^3e^y - 2x\sin y)dy = 0$$

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = 3x^2ye^y + 2\cos y \\ \frac{\partial u}{\partial y} = x^3e^yy + x^3e^y - 2x\sin y \end{cases} \Rightarrow u(x, y) = x^3ye^y + 2x\cos y + f(y)$$

$$\Rightarrow x^3e^yy + x^3e^y - 2x\sin y = \frac{\partial u}{\partial y} = x^3e^yy + x^3e^y - 2x\sin y + f'(y) \Rightarrow f'(y) = 0 \Rightarrow f(y) = D$$

Choose $D = 0$, then $u(x, y) = x^3ye^y + 2x\cos y$.

So the general solution to $(3x^2ye^y + 2\cos y)dx + (x^3e^yy + x^3e^y - 2x\sin y)dy = 0$ is:

$$u(x, y) = x^3ye^y + 2x\cos y = K,$$

where K is an arbitrary constant.

4. By Kirchhoff's laws,

Potential drop across L + Potential drop across R + Potential drop across $E = 0$,

that is, $5\frac{dI}{dt} + 10I + (-E) = 0$ or $\frac{dI}{dt} + 2I = \frac{E}{5}$.

- (a) $E = 40$, then $\frac{dI}{dt} + 2I = 8$. An integrating factor is $u(t) = e^{\int 2dt} = e^{2t}$.

$$e^{2t}I = \int 8e^{2t}dt = 4e^{2t} + c_1. \text{ With the initial condition } I(0) = 0, \text{ we have } c_1 = -4.$$

$$\therefore I = 4 - \frac{4}{e^{2t}} = 4(1 - e^{-2t})$$

- (b) $E = 20e^{-3t}$, then $\frac{dI}{dt} + 2I = 4e^{-3t}$. An integrating factor is $u(t) = e^{\int 2dt} = e^{2t}$.

$$e^{2t}I = \int 4e^{2t}e^{-3t}dt = 4\int e^{-t}dt = -4e^{-t} + c_1.$$

With the initial condition $I(0) = 0$, we have $c_1 = 4$.

$$\therefore I = -4e^{-3t} + \frac{4}{e^{2t}} = 4(e^{-2t} - e^{-3t})$$

- (c) $E = 50\sin 5t$, then $\frac{dI}{dt} + 2I = 10\sin 5t$. An integrating factor is $u(t) = e^{\int 2dt} = e^{2t}$.

$$e^{2t}I = 10\int e^{2t}\sin 5tdt \quad \text{using integration by parts} = 10\left(\frac{2e^{2t}\sin 5t}{29} - \frac{5e^{2t}\cos 5t}{29} + c_1\right).$$

With the initial condition $I(0) = 0$, $0 = -\frac{50}{29} + 10c_1 \Rightarrow c_1 = \frac{5}{29}$

$$\therefore I = \frac{20\sin 5t}{29} - \frac{50\cos 5t}{29} + \frac{50e^{-2t}}{29}$$

5. From $\frac{dA_1}{dt} = -\lambda_1 A_1$, $\int \frac{dA_1}{A_1} = \int -\lambda_1 dt$, then $\ln A_1 = -\lambda_1 t + c_1 \Rightarrow A_1 = e^{-\lambda_1 t + c_1}$

With the initial condition $A_1(0) = C$, we get $A_1(0) = C = e^{c_1}$.

$$\therefore A_1 = Ce^{-\lambda_1 t}.$$

Put $A_1 = Ce^{-\lambda_1 t}$ into $\frac{dA_2}{dt} = \lambda_1 A_1 - \lambda_2 A_2$, we have $\frac{dA_2}{dt} + \lambda_2 A_2 = C\lambda_1 e^{-\lambda_1 t}$

An integrating factor is $e^{\int \lambda_2 dt} = e^{\lambda_2 t}$.

So $e^{\lambda_2 t} A_2 = C\lambda_1 \int e^{(\lambda_2 - \lambda_1)t} dt = \frac{C\lambda_1}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} + c_1$

With the initial condition $A_2(0) = 0$, we get $A_2(0) = \frac{C\lambda_1}{\lambda_2 - \lambda_1} + c_1 = 0 \Rightarrow c_1 = -\frac{C\lambda_1}{\lambda_2 - \lambda_1}$.

$$\therefore A_2 = \frac{C\lambda_1}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}).$$

Put $A_2 = \frac{C\lambda_1}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t})$ into $\frac{dA_3}{dt} = \lambda_2 A_2$, we have

$$\frac{dA_3}{dt} = \frac{C\lambda_1\lambda_2}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

$$\Rightarrow A_3 = \frac{C\lambda_1\lambda_2}{\lambda_2 - \lambda_1} \left(\int e^{-\lambda_1 t} dt - \int e^{-\lambda_2 t} dt \right) = \frac{C\lambda_1\lambda_2}{\lambda_2 - \lambda_1} \left(-\frac{1}{\lambda_1} e^{-\lambda_1 t} + \frac{1}{\lambda_2} e^{-\lambda_2 t} \right) + c_1$$

With the initial condition $A_3(0) = 0$, we have $0 = A_3(0) = \frac{C\lambda_1\lambda_2}{\lambda_2 - \lambda_1} \left(-\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) + c_1 \Rightarrow c_1 = C$.

$$\therefore A_3 = \frac{C\lambda_1\lambda_2}{\lambda_2 - \lambda_1} \left(-\frac{1}{\lambda_1} e^{-\lambda_1 t} + \frac{1}{\lambda_2} e^{-\lambda_2 t} \right) + C$$

6. (a) $y = 1 + \frac{1}{z} \Rightarrow \frac{dy}{dx} = -\frac{1}{z^2} \frac{dz}{dx}$.

$$\frac{dy}{dx} = \frac{1}{x} y^2 + \frac{1}{x} y - \frac{2}{x} \underset{\frac{dy}{dx} = -\frac{1}{z^2} \frac{dz}{dx}}{\Rightarrow} -\frac{1}{z^2} \frac{dz}{dx} = \frac{1}{x} \left(1 + \frac{1}{z} \right)^2 + \frac{1}{x} \left(1 + \frac{1}{z} \right) - \frac{2}{x}$$

$$\Rightarrow -\frac{1}{z^2} \frac{dz}{dx} = \frac{1}{x} \left(1 + \frac{2}{z} + \frac{1}{z^2} \right) + \frac{1}{x} \left(1 + \frac{1}{z} \right) - \frac{2}{x} = \frac{1}{x} \frac{3}{z} + \frac{1}{x} \frac{1}{z^2}$$

$$\Rightarrow \frac{dz}{dx} + \frac{3}{x} z = -\frac{1}{x}$$

- (b) An integrating factor is $e^{\int \frac{3}{x} dx} = e^{3 \ln x} = e^{\ln(x^3)} = x^3$. Then

$$\frac{dz}{dx} + \frac{3}{x} z = -\frac{1}{x} \Rightarrow \frac{d(x^3 z)}{dx} = -x^2 \Rightarrow x^3 z = -\frac{x^3}{3} + C$$

$$\Rightarrow z = -\frac{1}{3} + \frac{C}{x^3} = \frac{-x^3 + 3C}{3x^3} \Rightarrow y = 1 + \frac{1}{z} = \frac{2x^3 + 3C}{-x^3 + 3C} \underset{3C=K}{=} \frac{K + 2x^3}{K - x^3}$$

-End-