
EE3121

Differential Equations for Engineers

Tutorial 9a



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Tutorial

Problem:

Consider the heated rod example analyzed in Section 9.1.2. A rod of length L subject to uniform heating across it and to both ends being held at zero temperature will have the steady-state temperature distribution

$$\frac{4T_0}{L^2}x(L - x)$$

See Figure 9.16.

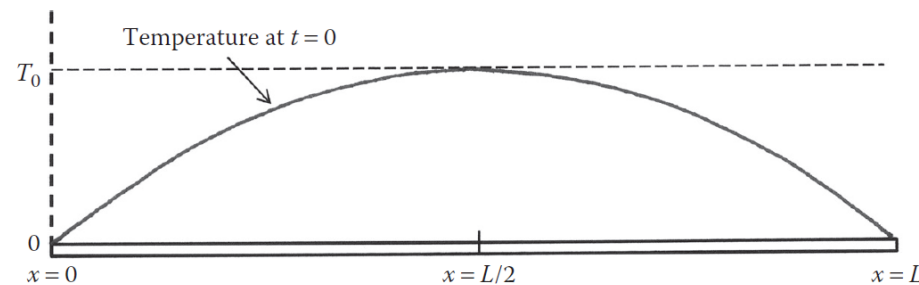


FIGURE 9.16 Temperature distribution across a uniformly heated rod.

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Use a Fourier series expansion to find the temperature as a function of space and time if the heating is removed. The equations to be solved are

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

$$T(0, t) = 0, \quad T(L, t) = 0, \quad T(x, 0) = \frac{4T_0}{L^2}x(L - x).$$

Check your results by evaluating your series solution for $T(L/2, 0)$ and comparing it to the given initial value T_0 . If you need to evaluate the sum of an infinite series use web resources.

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Solution:

Consider the trial solution $T(x, t) = \Phi(x)\Theta(t)$, which gives

$$\frac{\partial^2 T}{\partial x^2} = \frac{d^2 \Phi(x)}{dx^2} \Theta(t) = \frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\Phi(x)}{\alpha} \frac{d\Theta(t)}{dt}$$

$$\frac{1}{\Phi(x)} \frac{d^2 \Phi(x)}{dx^2} = \frac{1}{\alpha \Theta(t)} \frac{d\Theta(t)}{dt} = -\omega^2$$

$$\frac{d^2 \Phi(x)}{dx^2} + \omega^2 \Phi(x) = 0, \quad \frac{d\Theta(t)}{dt} + \alpha \omega^2 \Theta(t) = 0.$$

$$\Phi(x) = c_1 \cos \omega x + c_2 \sin \omega x, \quad \Theta(t) = c_3 \exp(-\alpha \omega^2 t).$$

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Boundary conditions $T(0, t) = 0, T(L, t) = 0$

$$\Phi(0) = c_1 = 0, \quad \Phi(L) = c_1 \cos \omega L + c_2 \sin \omega L = 0.$$

$$\sin \omega L = 0 \quad \Rightarrow \quad \omega = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots$$

$$T(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \exp\left(-\alpha\left(\frac{n\pi}{L}\right)^2 t\right).$$

Initial condition $T(x, 0) = \frac{4T_0}{L^2}x(L - x)$

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) = \frac{4T_0}{L^2}x(L - x).$$

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$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) = \frac{4T_0}{L^2}x(L-x)$$

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} 0 & n \neq m \\ L/2 & n = m \end{cases}$$

$$c_n = \frac{2}{L} \int_0^L \frac{4T_0}{L^2}x(L-x) \sin \frac{n\pi x}{L} dx = \frac{8T_0}{L^3} \int_0^L x(L-x) \sin \frac{n\pi x}{L} dx$$

Use the integration by parts to compute the integral.

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Check the result at the point $x = L/2, t = 0$

$$T\left(\frac{L}{2}, 0\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{2}\right) = \sum_{k=0}^{\infty} (-1)^k c_{2k+1}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}$$

END