
EE1001

Foundations of Digital Techniques and Data Analysis

Essence of Optimization – Lagrange Multipliers

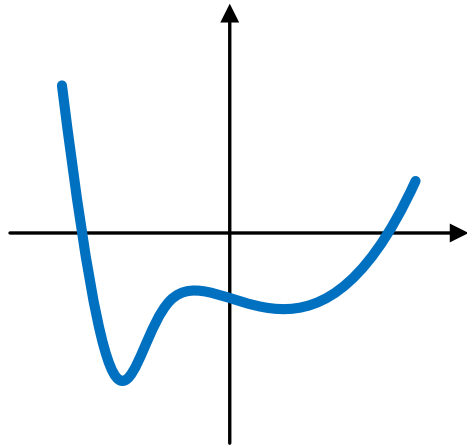


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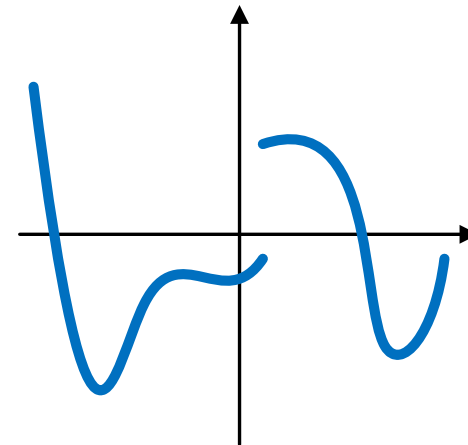
Preliminaries

- Continuous function
- Smooth function
- Minimum, local minimum, global minimum
- Maximum, local Maximum, global Maximum
- Convex function
- Slope and derivative
- Derivative and minimum/maximum

Continuous Function



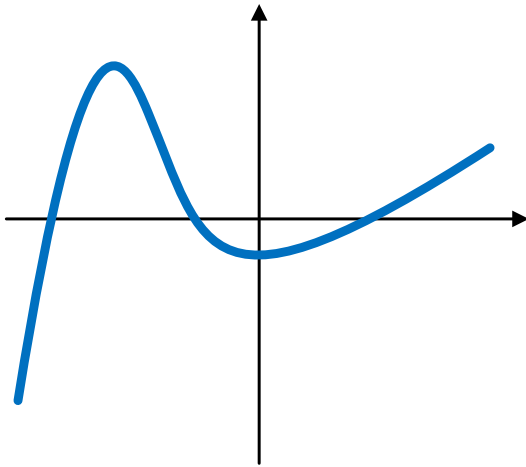
A continuous function



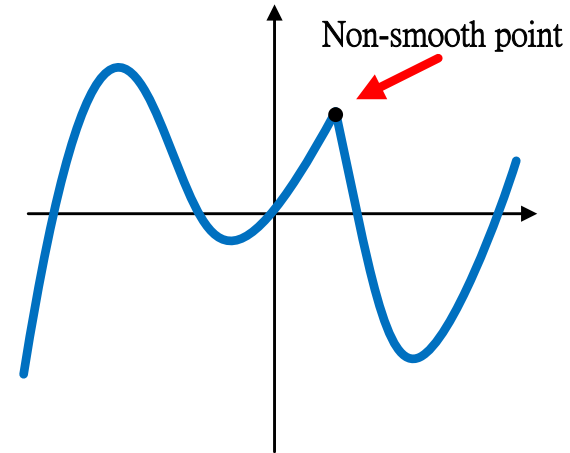
A discontinuous function

- A function is continuous if its graph is a single unbroken curve that you could draw without lifting your pen from the paper
- Not a formal definition

Smooth Function



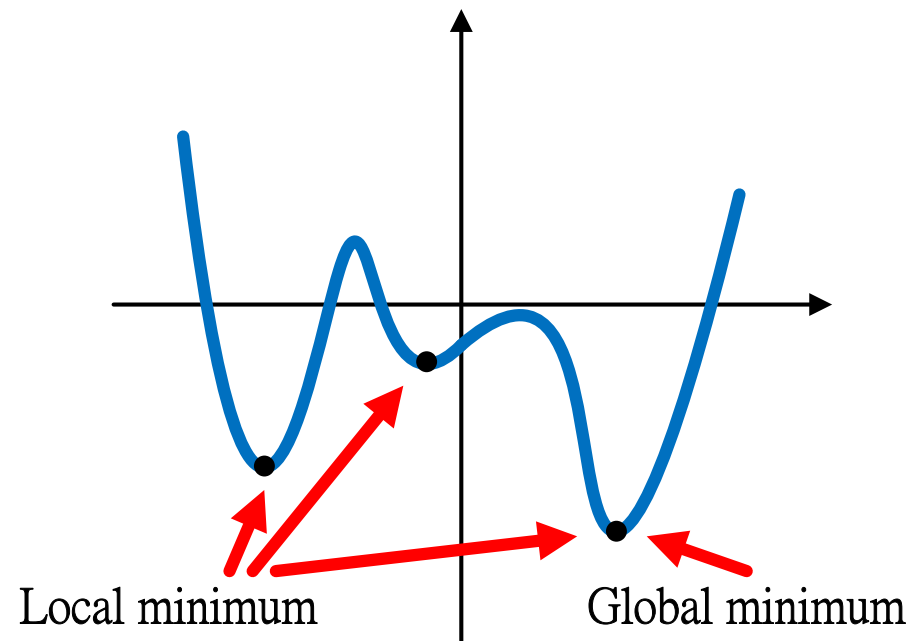
A smooth function



A non-smooth function

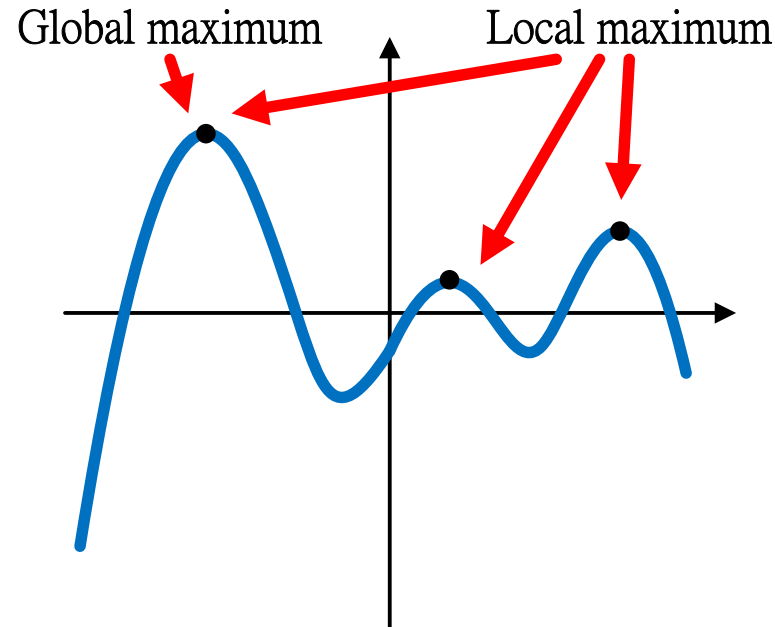
- If a continuous function has a curve free from irregularities, roughness, or projections
- Not a formal definition

Local Minimum vs. Global Minimum



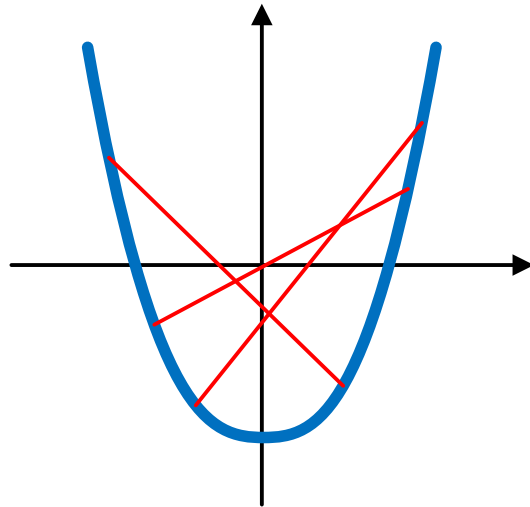
- Local minimum: a minimum within its neighborhood
- Global minimum: the smallest value of a function over its entire range
- A global minimum is of course also a local minimum

Local Maximum vs. Global Maximum

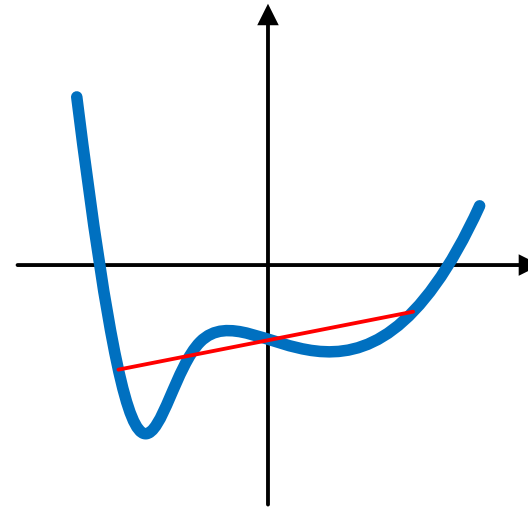


- Local maximum: a maximum within its neighborhood
- Global maximum: the largest value of a function over its entire range
- A global maximum is of course also a local maximum

Convex Function



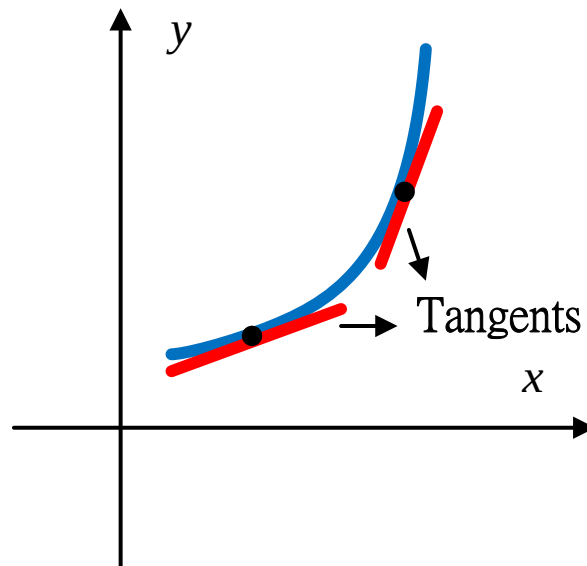
Convex function



Non-convex function

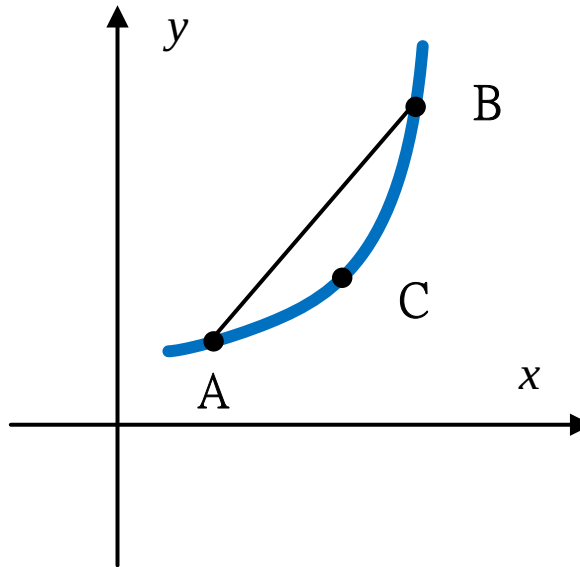
- The line drawn between any two points on the curve stays above the curve and will not cross over the curve

Convex Function



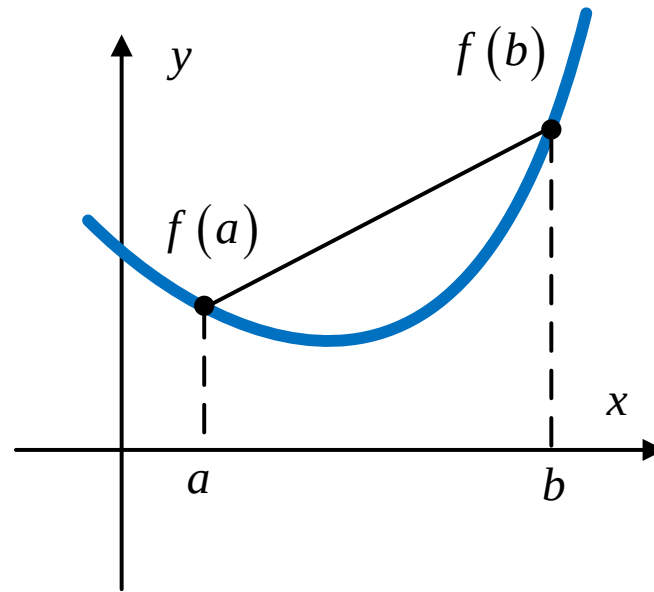
- A function is convex if f is above all its tangents

Convex Function



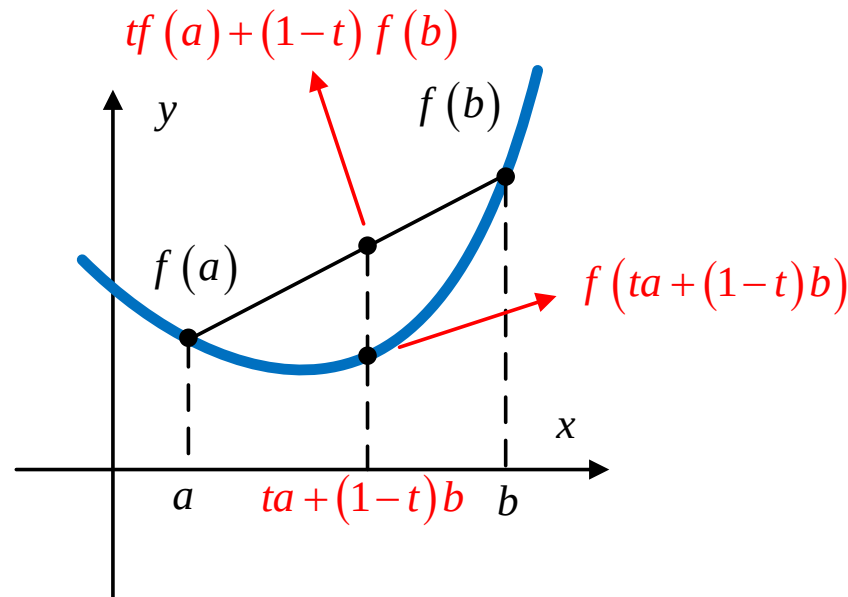
- A function is convex if for any two points A and B , and for $A < C < B$, $f(C)$ lies below the segment $[f(A), f(B)]$

Convex Function: Formula



- Any x value between a and b is a function of t from 0 to 1: $x = ta + (1 - t)b$
- If $t = 0$, $x = 0a + 1b = b$; if $t = 1$, $x = 1a + 0b = a$
- If t is between 0 and 1, we get values between a and b

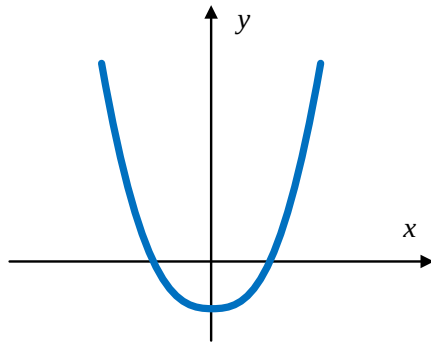
Convex Function: Formula



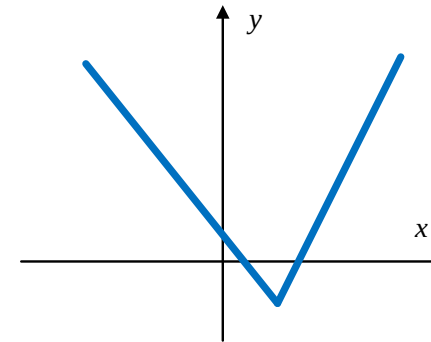
- The curve is at $y = f(ta + (1-t)b)$ at $x = ta + (1-t)b$
- The line is at $y = tf(a) + (1-t)f(b)$ at $x = ta + (1-t)b$
- The function is convex if:

$$tf(a) + (1-t)f(b) \geq f(ta + (1-t)b)$$

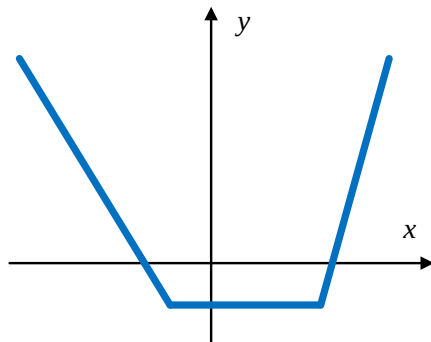
Examples of Convex Functions



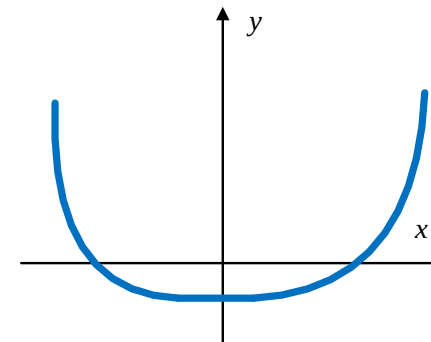
Smooth



Non-smooth

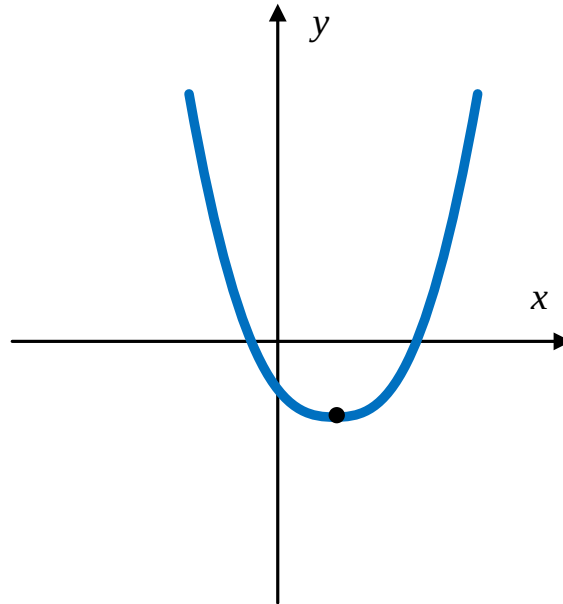


Non-smooth



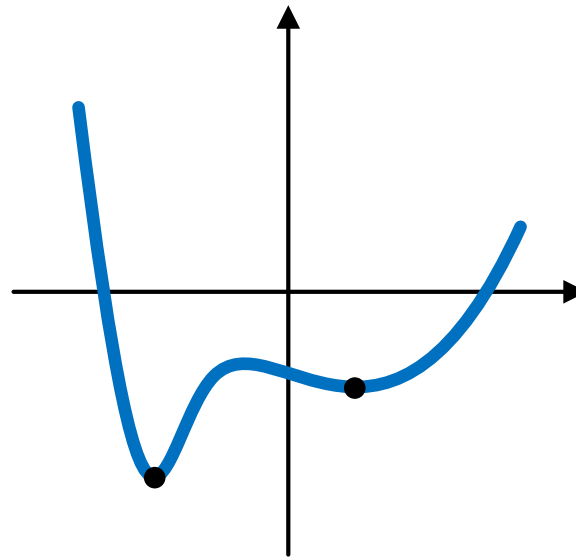
Smooth

Minimums of Convex Functions



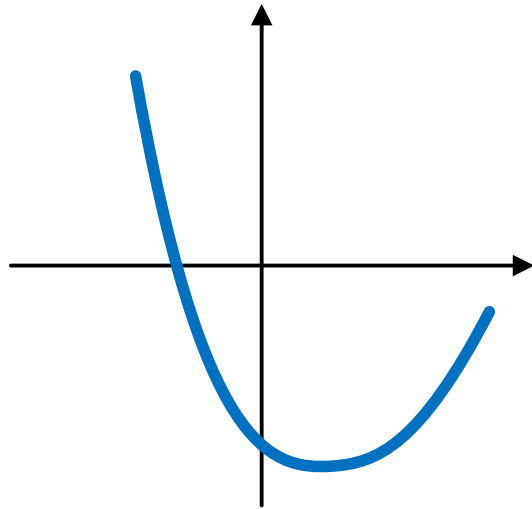
- A convex function has only one minimum, which is the global minimum

Minimums of Non-convex Functions

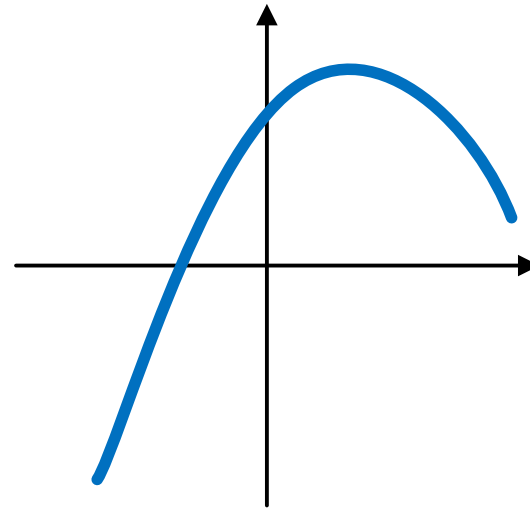


- A non-convex function may have more than one minimum, and thus may have local minimums

Concave Function



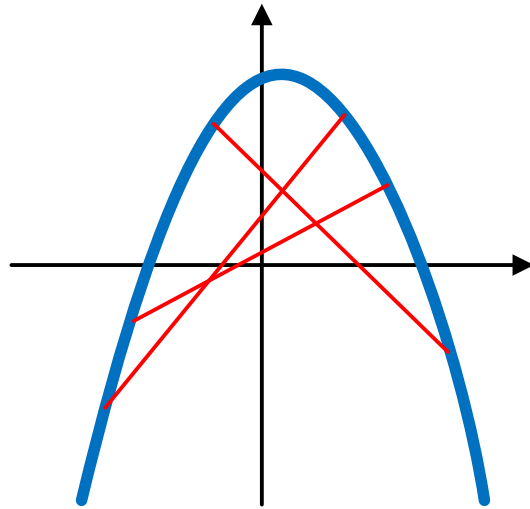
Convex function



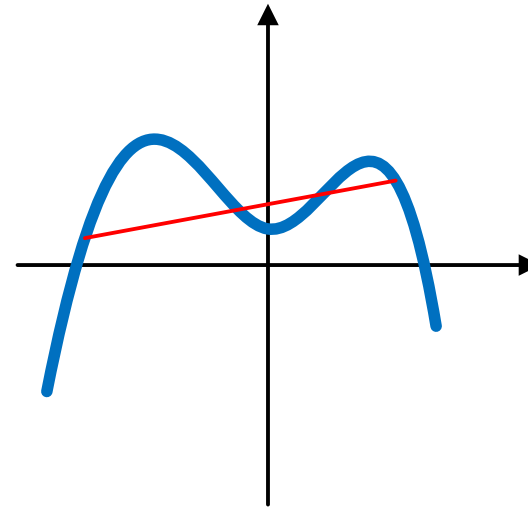
A concave function

- A convex function upside down

Concave Function



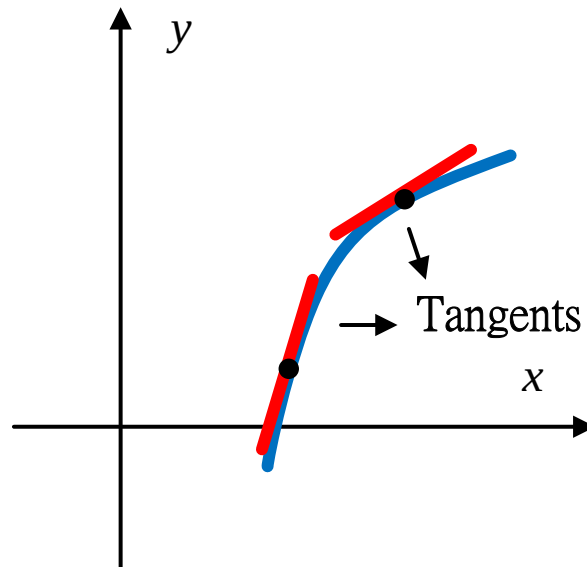
Concave function



Non-concave function

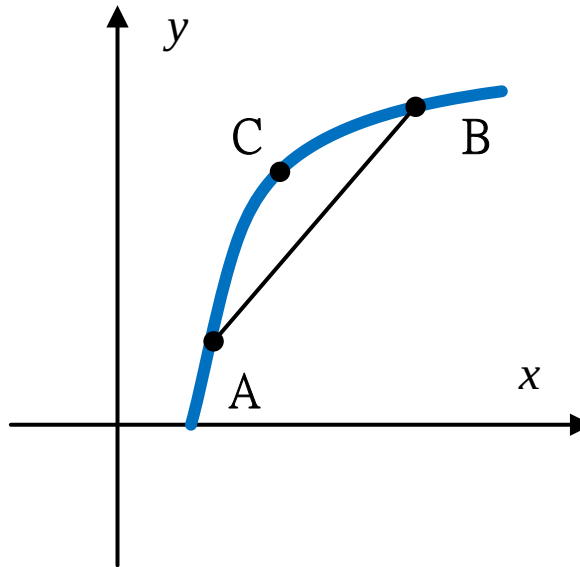
- The line drawn between any two points on the curve stays below the curve and will not cross over the curve

Concave Function



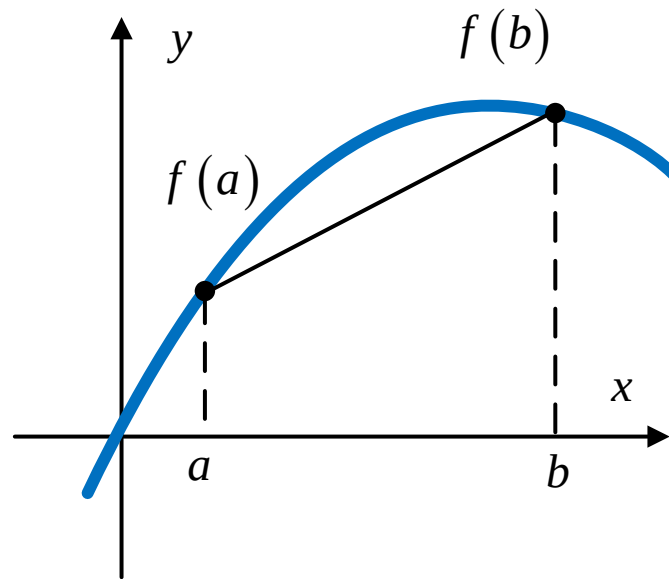
- A function is concave if f is below all its tangents

Concave Function



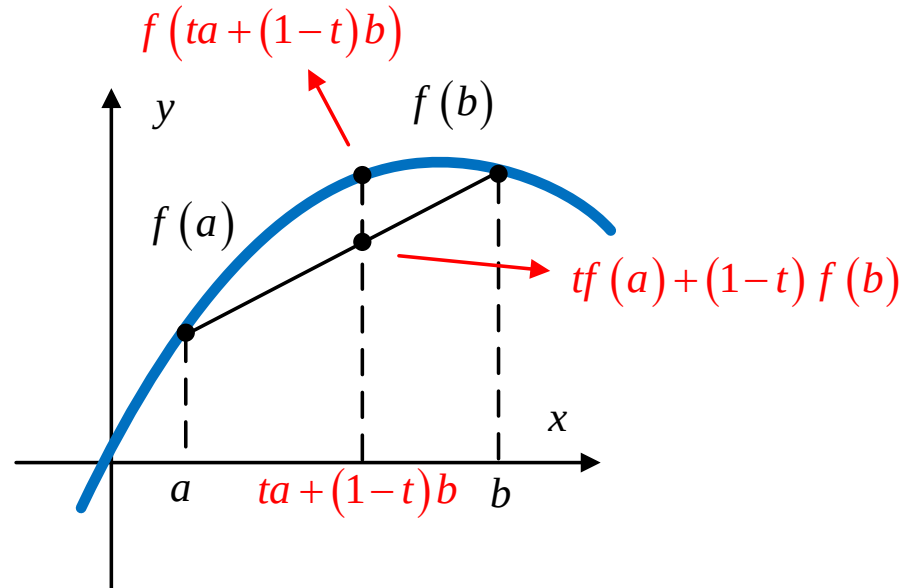
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Concave Function: Formula



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- If $t = 0$, $x = 0a + 1b = b$; if $t = 1$, $x = 1a + 0b = a$
- If t is between 0 and 1, we get values between a and b

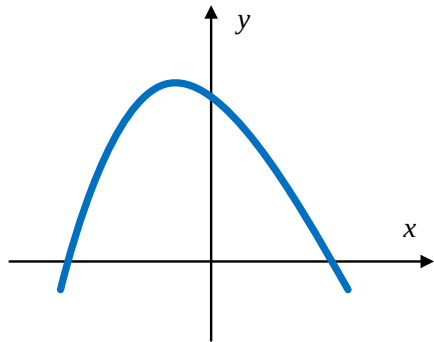
Concave Function: Formula



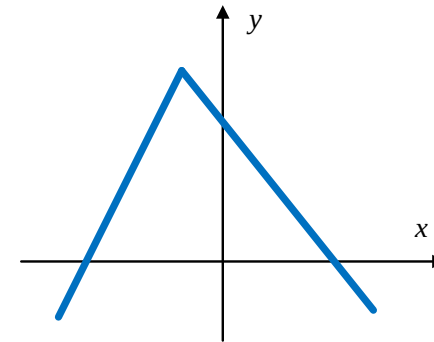
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- The line is at $y = tf(a) + (1-t)f(b)$ at $x = ta + (1-t)b$
- The function is concave if:

$$tf(a) + (1-t)f(b) \leq f(ta + (1-t)b)$$

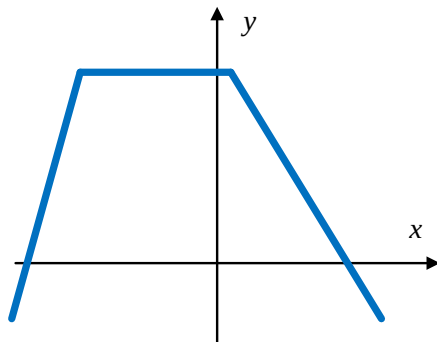
Examples of Convex Functions



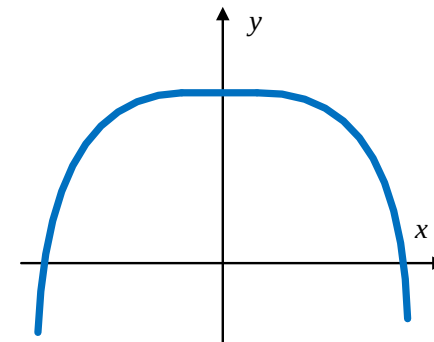
Smooth



Non-smooth

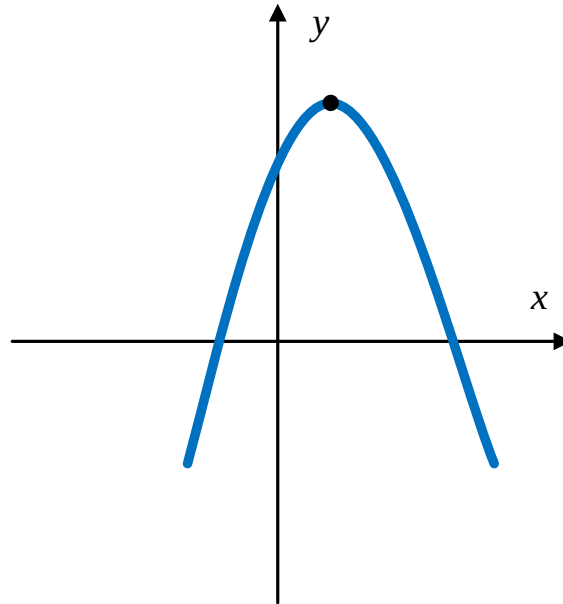


Non-smooth



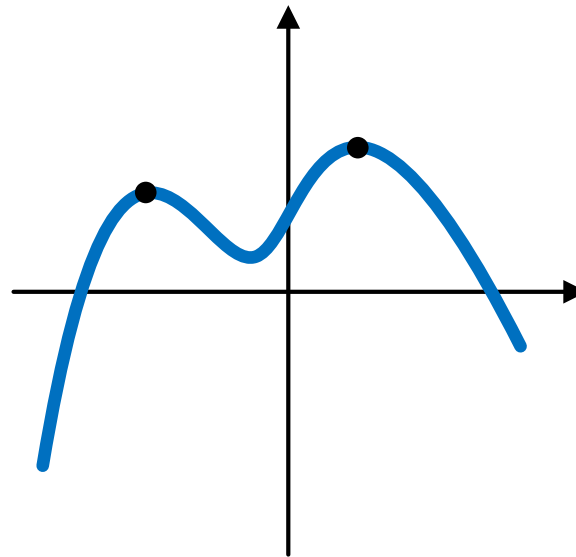
Smooth

Maximums of Concave Functions



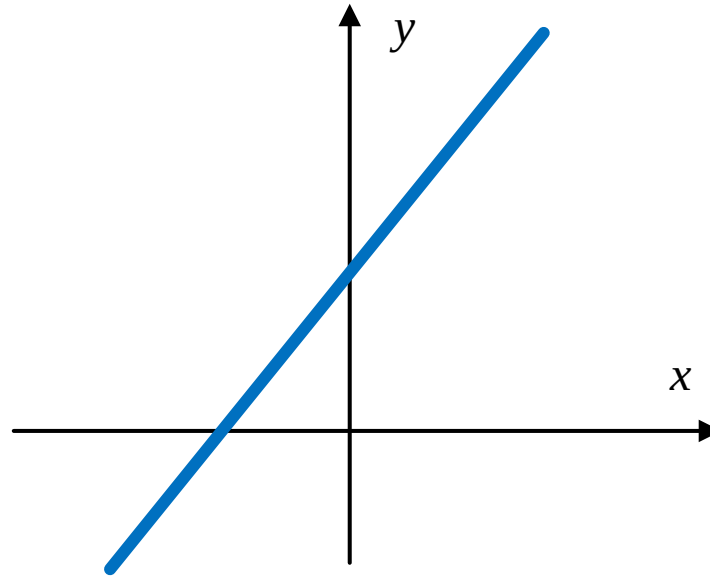
- A concave function has only one maximum, which is the global maximum

Maximums of Non-concave Functions



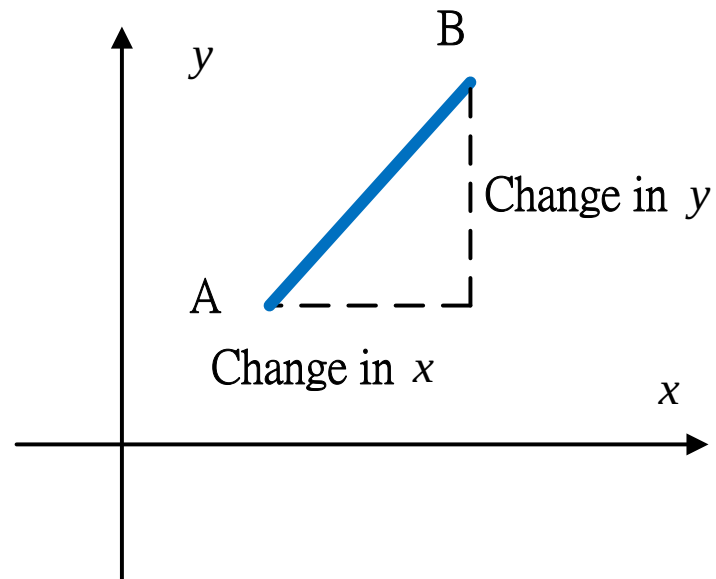
- A non-concave function may have more than one maximum, and thus may have local maximums

Affine Function



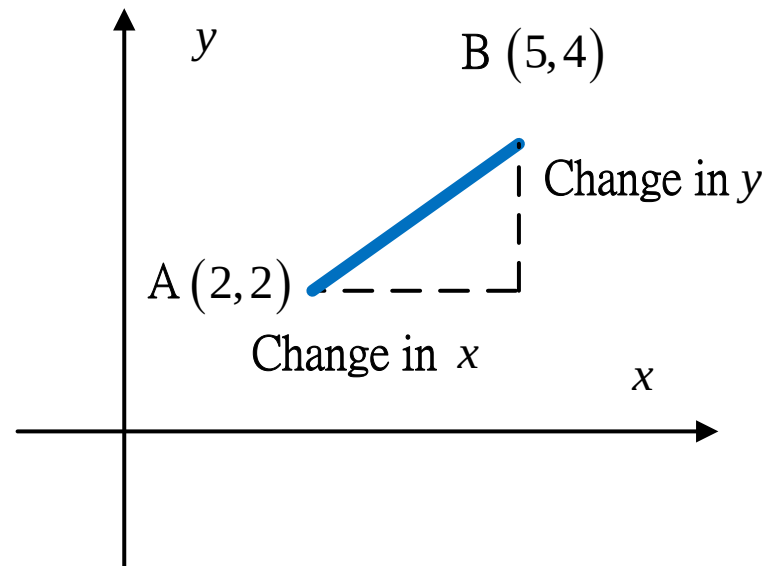
- Affine function: $y = ax + b$
- Both onvex and concave
- This is why Linear Programming problems are easy

Slope



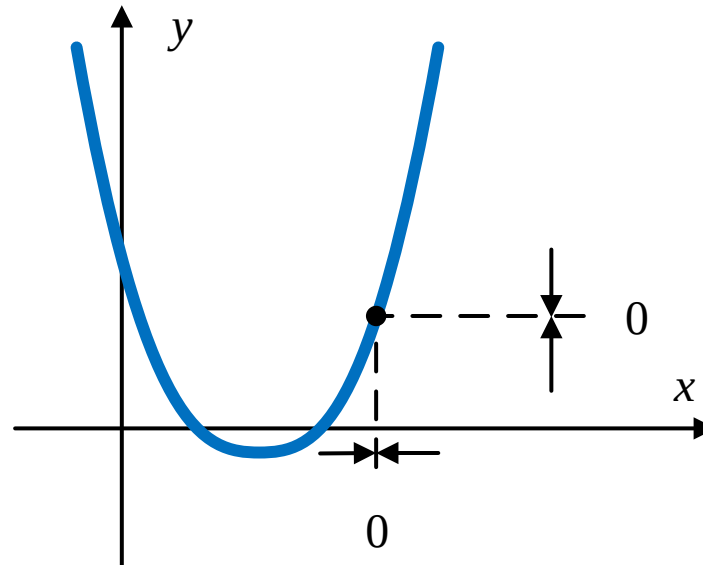
- Slope between two points A and B : $\frac{\text{Change in } y}{\text{Change in } x}$

Slope



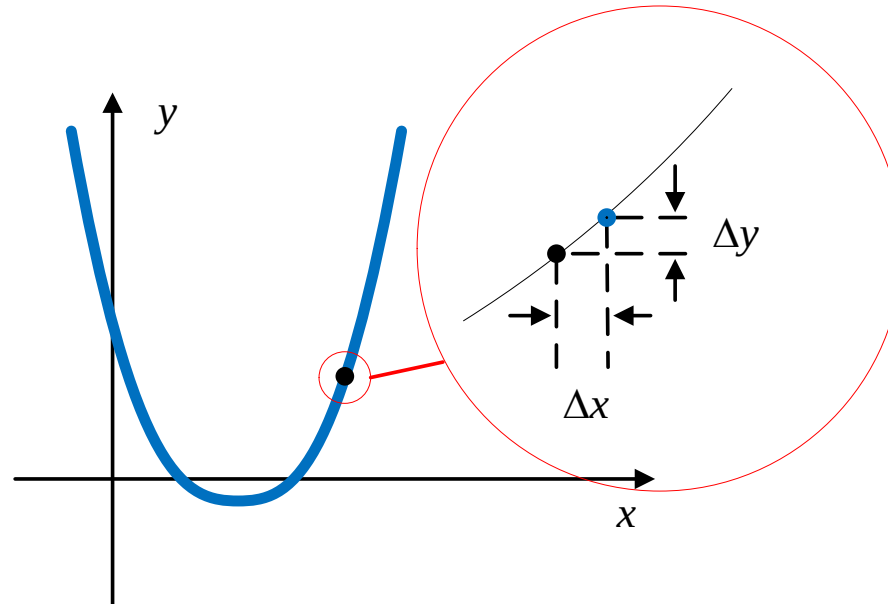
- Slope between two points A and B : $\frac{\text{Change in } y}{\text{Change in } x} = \frac{5-2}{4-2} = 1.5$

Slope at a Point



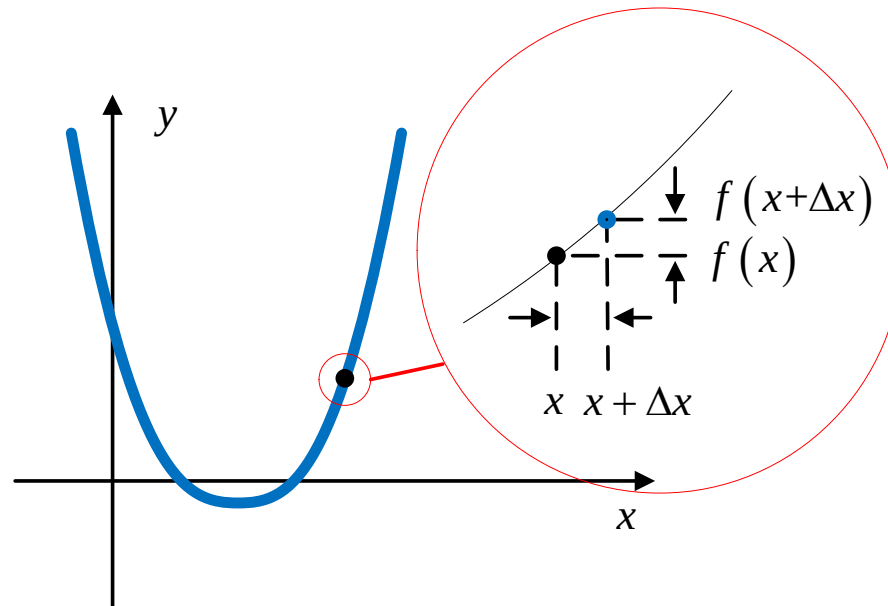
- Slope at a point? There is nothing to measure!

Slope at a Point



- Use a small difference, and then have it shrink towards zero
- Δx and Δy are very small

Slope at a Point: Derivative



- If $y = f(x)$, then as x changes from x to $x + \Delta x$, y changes from $f(x)$ to $f(x + \Delta x)$

Derivative

- Definition of derivative at x (by filling in this slope formula):

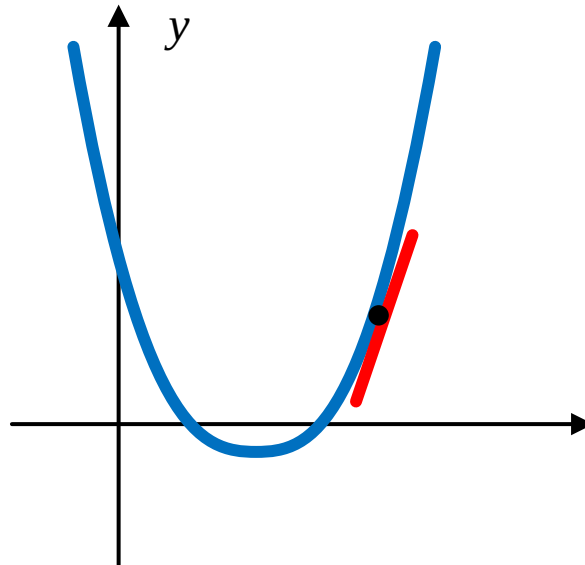
$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

and then make Δx shrink towards zero

- Denoted as

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

Derivative is also slope of the tangent line



- The derivative at a certain point is also the slope of the tangent line at that point

Example 1

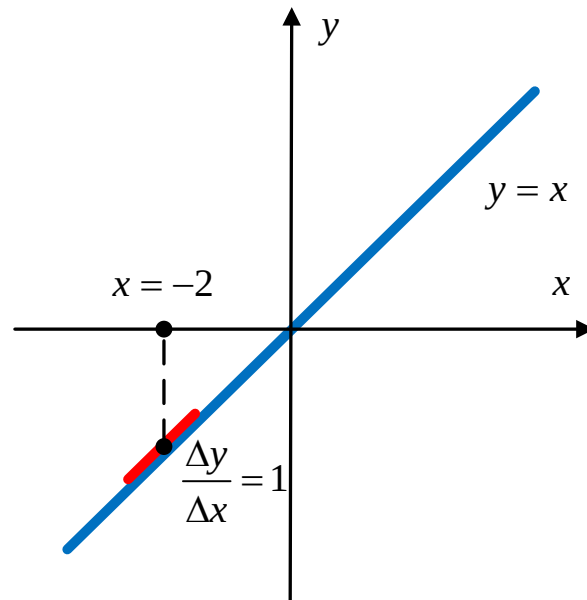
$$f(x) = ax$$

- If $y = f(x) = ax$, then as x changes from x to $x + \Delta x$, y changes from $f(x) = ax$ to $f(x + \Delta x) = a(x + \Delta x) = ax + a\Delta x$

- Derivative at x :

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{ax + a\Delta x - ax}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{a\Delta x}{\Delta x} \\ &= a\end{aligned}$$

Example 1



■ If $y = x$, then

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = 1$$

Example 2

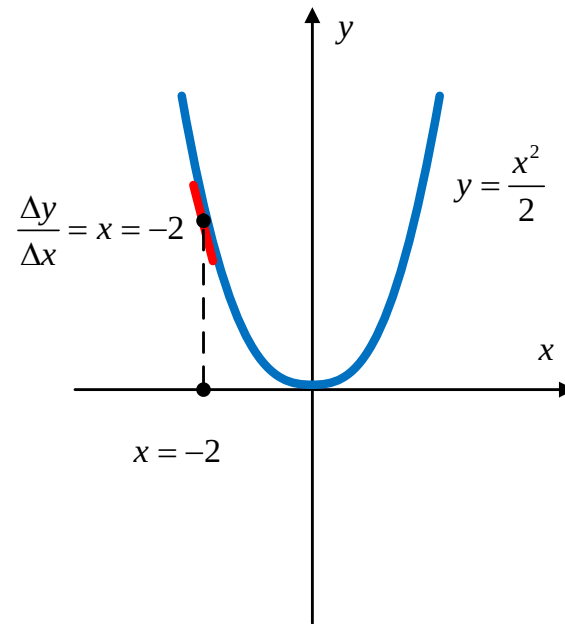
$$f(x) = bx^2$$

■ If $y = f(x) = bx^2$, then as x changes from x to $x + \Delta x$, y changes from $f(x) = bx^2$ to $f(x + \Delta x) = b(x + \Delta x)^2 = bx^2 + 2bx\Delta x + b(\Delta x)^2$

■ Derivative at x :

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{bx^2 + 2bx\Delta x + b(\Delta x)^2 - bx^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2bx\Delta x + b(\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2bx + b\Delta x) = 2bx\end{aligned}$$

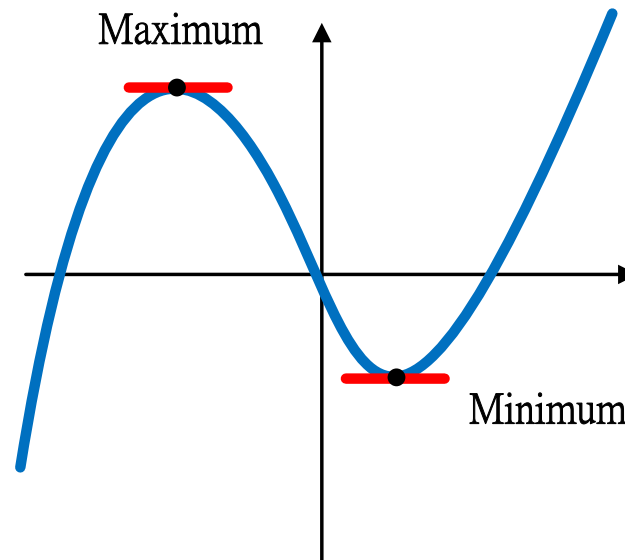
Example 2



■ If $y = x^2$, then

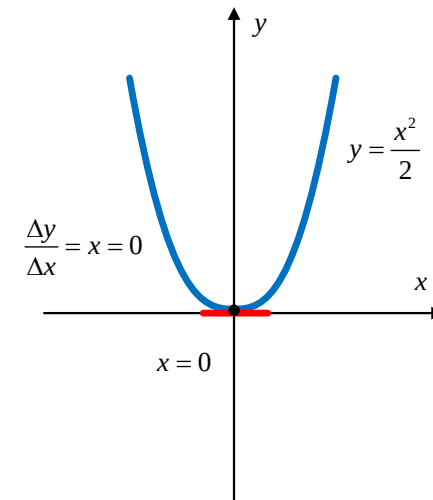
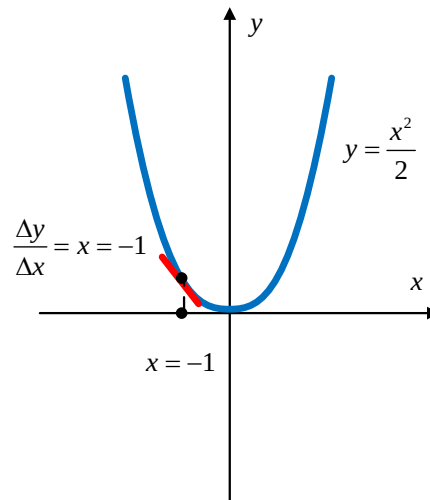
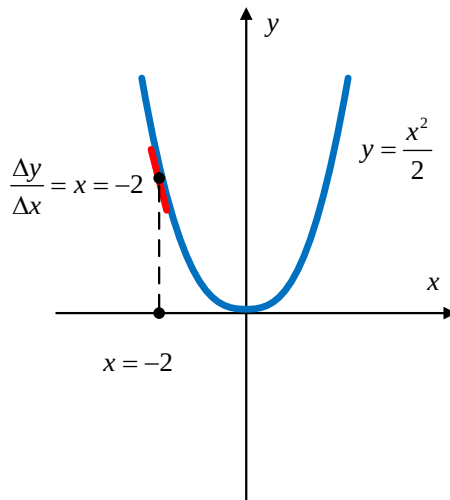
$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = 2x$$

Maximum, Minimum



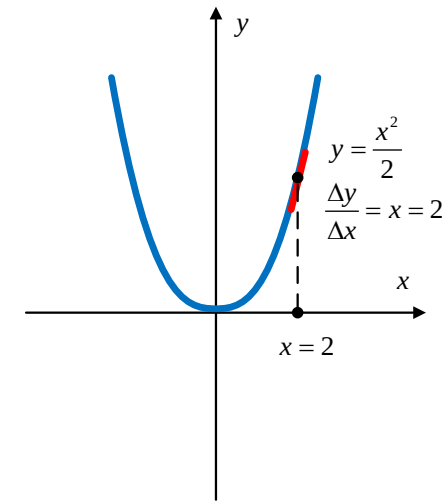
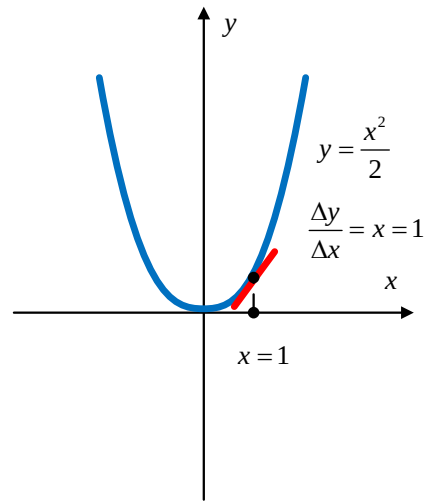
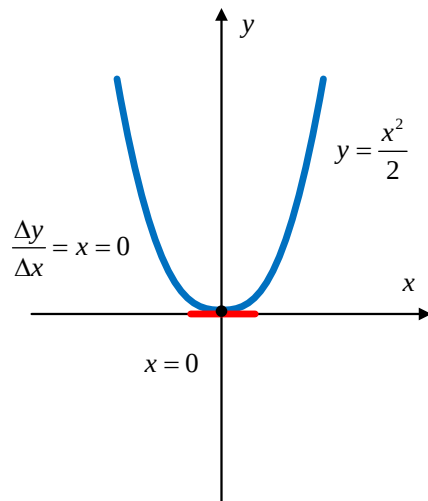
- In a smooth function, a minimum or maximum are where the function flattens out
- The slope/derivative is zero
- If $y = f(x)$, then as x changes from x to $x + \Delta x$, y stays unchanged

Minimum



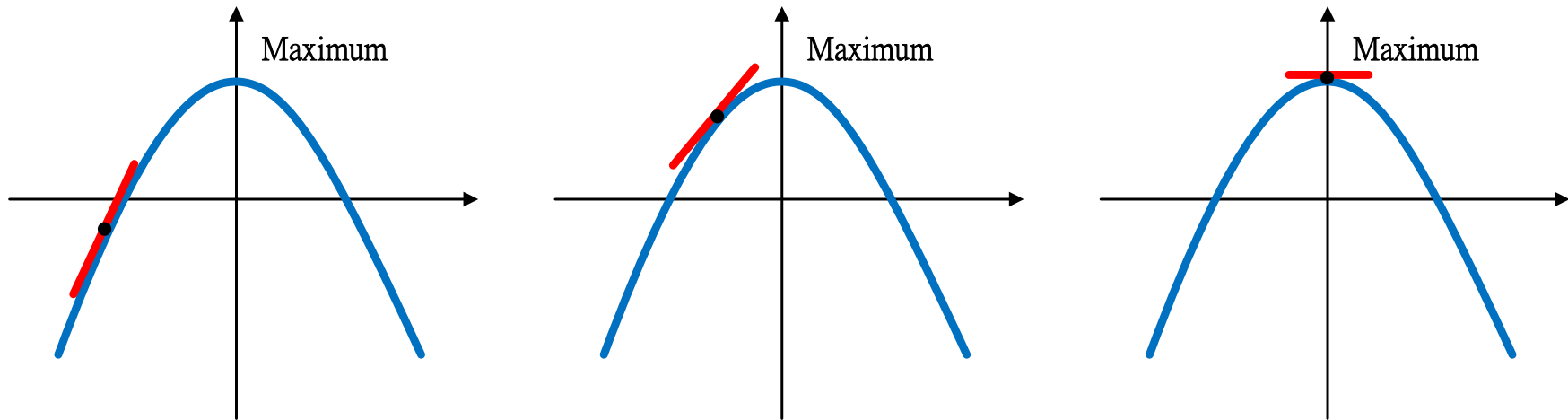
- The slope/derivative is negative before (on the left)
- As it approaches the minimum (from the left), the slope/derivative increases, finally to zero
- In other words, the slope/derivative gets larger and goes through zero at the minimum point

Minimum



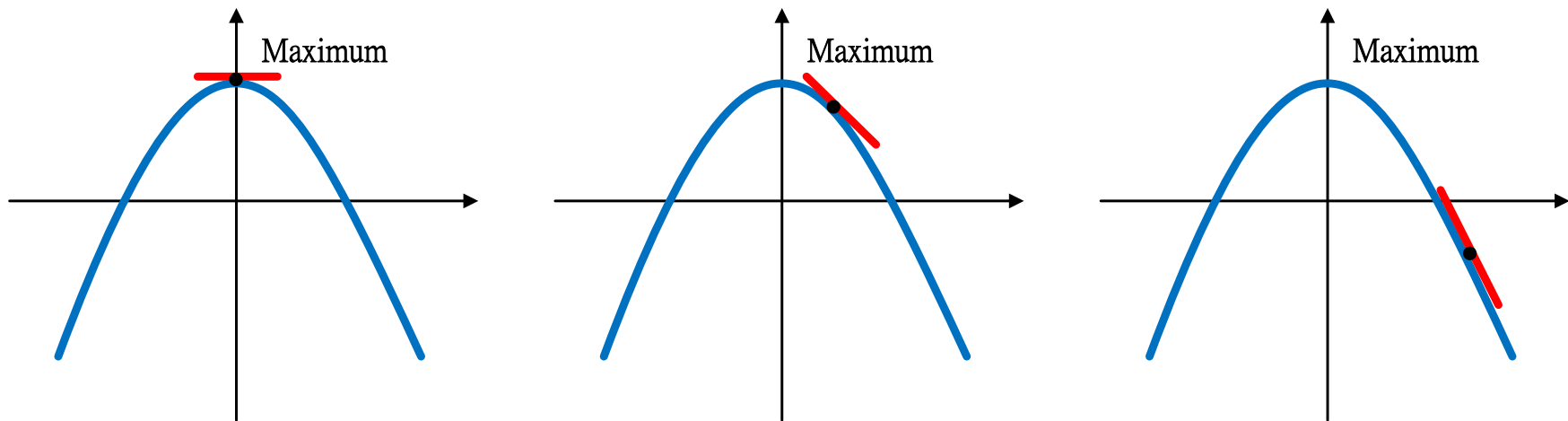
- The slope/derivative is positive after (on the right)
- As it leaves the minimum (to the right), the slope/derivative also increases

Maximum



- The slope/derivative is positive before (on the left)
- As it approaches the maximum (from the left), the slope/derivative decreases, finally to zero
- In other words, the slope/derivative gets smaller and goes through zero at the maximum point

Maximum

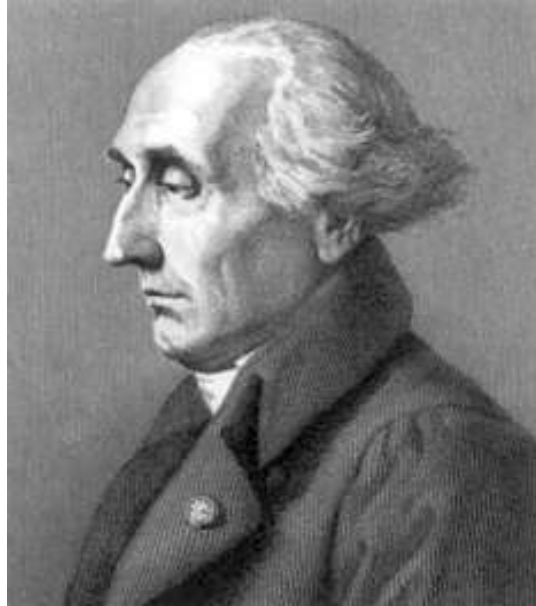


- The slope/derivative is negative after (on the right)
- As it leaves the maximum (to the right), the slope/derivative also decreases

Lagrange Multipliers

- The Milkmaid Problem
- Lagrange Multipliers
- Examples

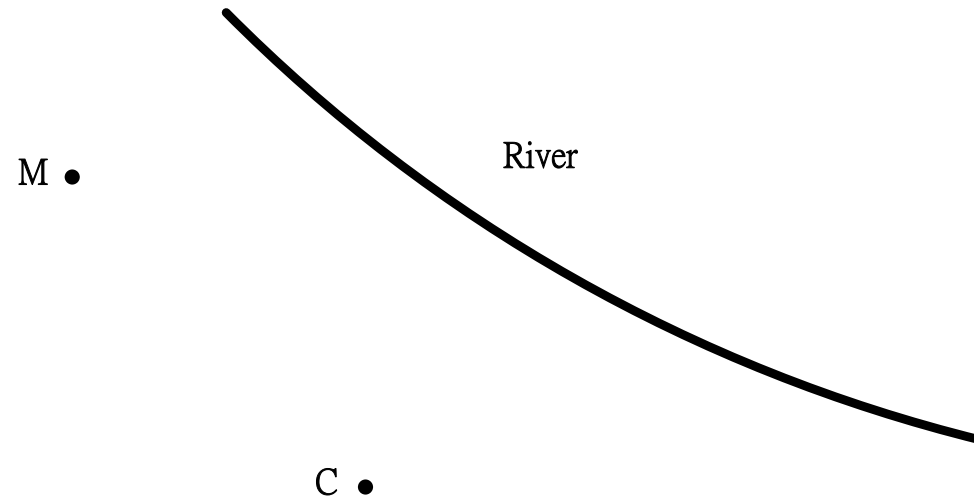
Lagrange and Lagrange Multipliers



Joseph-Louis Lagrange

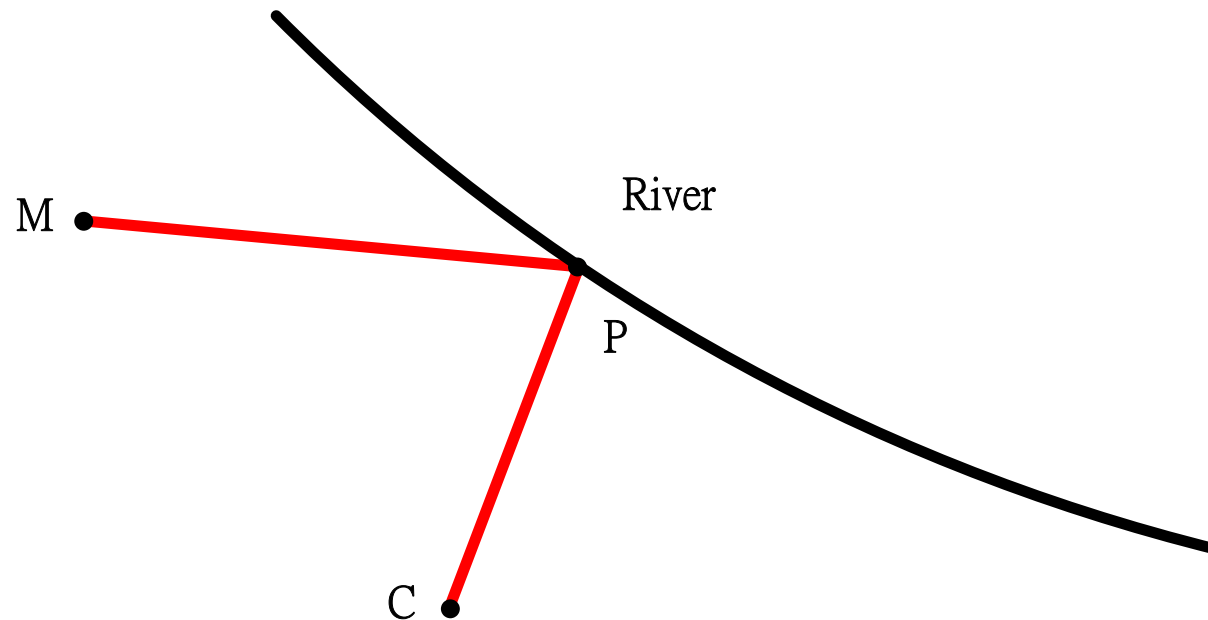
- Lagrange was one of the creators of the calculus of variations, deriving the Euler-Lagrange equations for extrema of functionals
- He also extended the method to take into account possible constraints, arriving at the method of Lagrange multipliers
- In his book: *Leons sur le calcul des fonctions*, 1904

The Milkmaid Problem



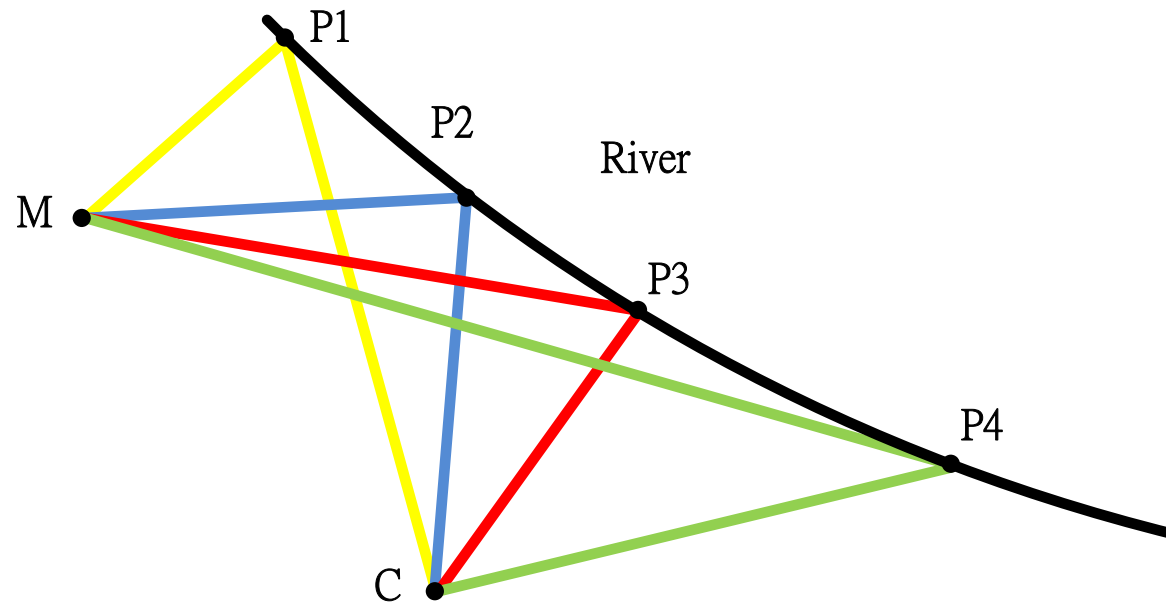
- The milkmaid is at point M , the cow is at point C
- Before gathering the milk, she has to rinse out her bucket in the river
- Field is flat and uniform, all points on the river bank are equally good
- She's in a hurry to get back for a date with a handsome young goatherd

Shortest Path



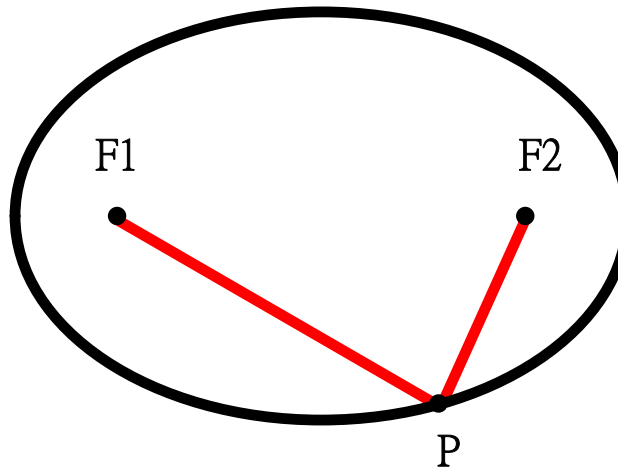
- What is the best point P on the riverbank for her to rinse the bucket?
- What is the shortest path $MP + PC$?
- Your guesses? Why?

Let Us Have a Vote



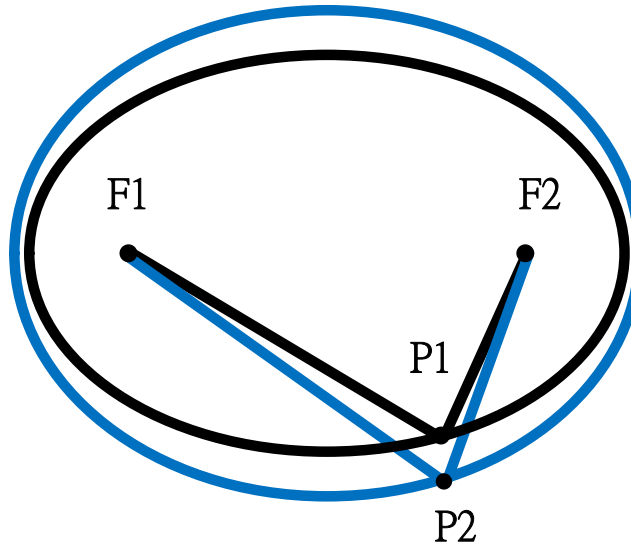
- Which points are possible?
- Which points are impossible?

We Need Mathematics



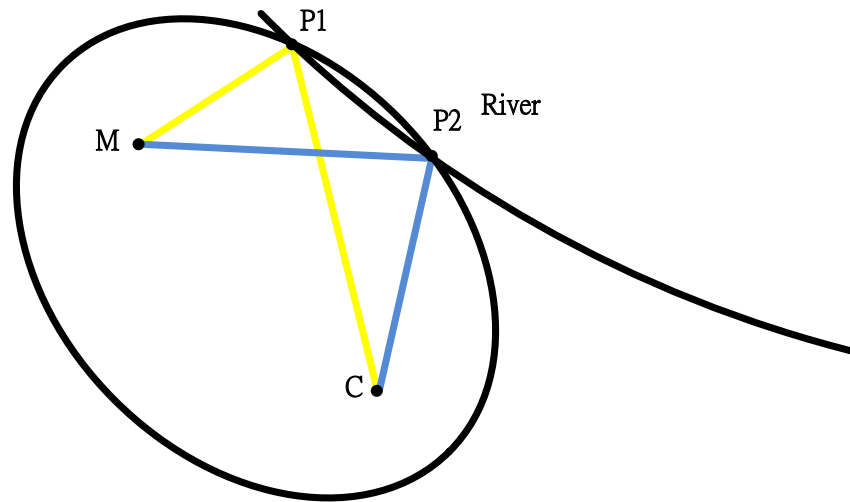
- Given an ellipse
- $F1, F2$: foci of the ellipse
- For every point P on the ellipse, the distance from P to one focus of the ellipse plus that to the other focus is exactly the same
- $F1P + PF2$ is a constant

Properties of Ellipses



- Consider two ellipses, one with a smaller size and the other with a larger size
- The foci of both ellipses are the same
- $F1P1 + P1F2$ is a constant, $F1P2 + P2F2$ is also a constant
- $F1P1 + P1F2 < F1P2 + P2F2$

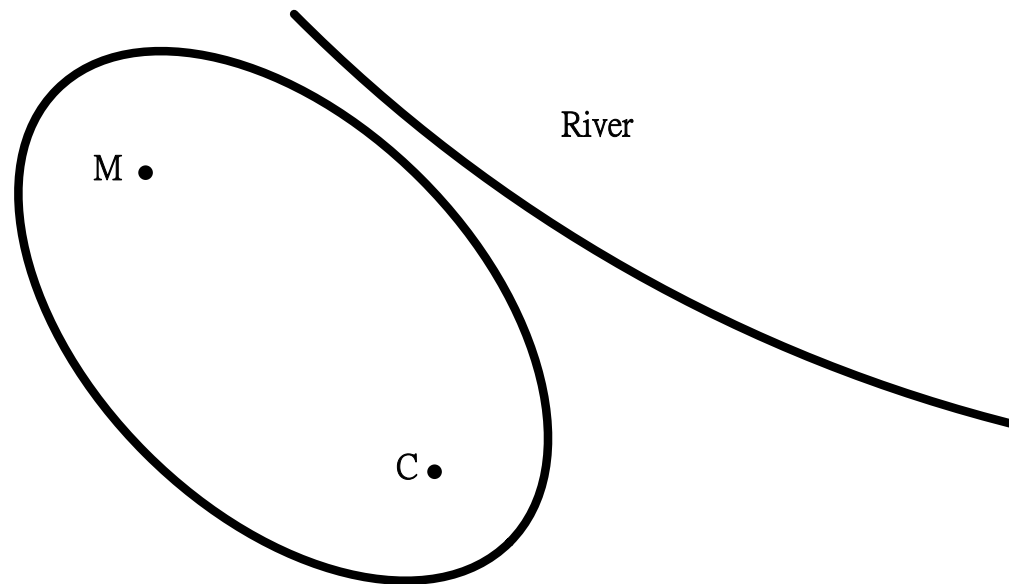
Intersecting the Ellipse and the River



- Pick M and C as the foci, and draw an ellipse
- Two intersecting points: P_1 and P_2
- $MP_1 + P_1C = MP_2 + P_2C$

May Have No Intersection

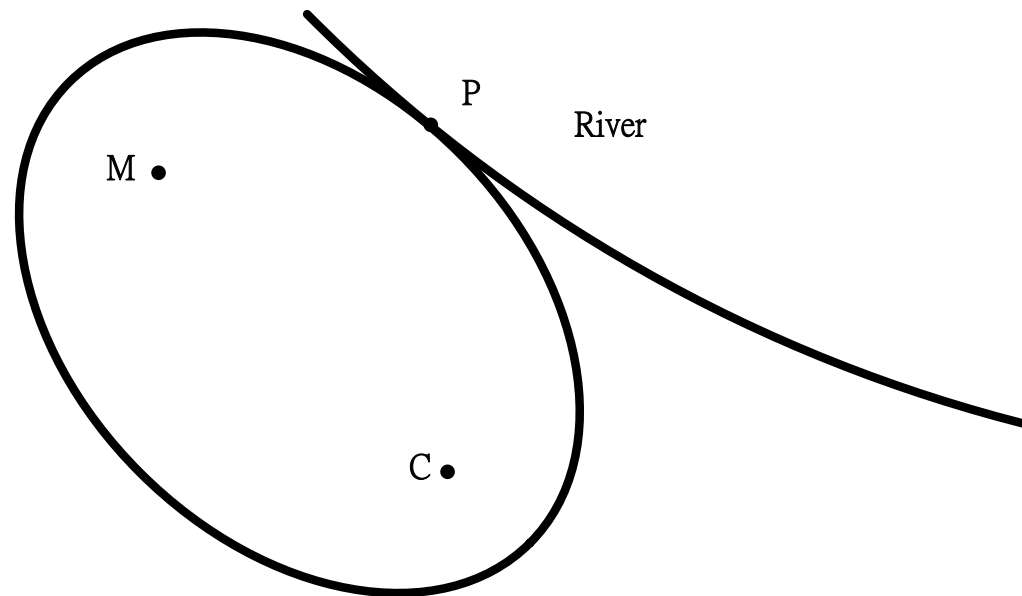
- The size of ellipse can still be made smaller



- But when it is too small, no intersection

Optimal P

- To find the smallest ellipse that intersects the curve of the river

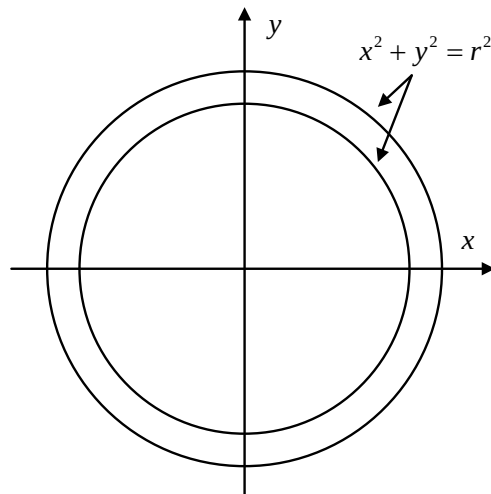


- Perfect ellipse: one that is just **TANGENT** to the riverbank

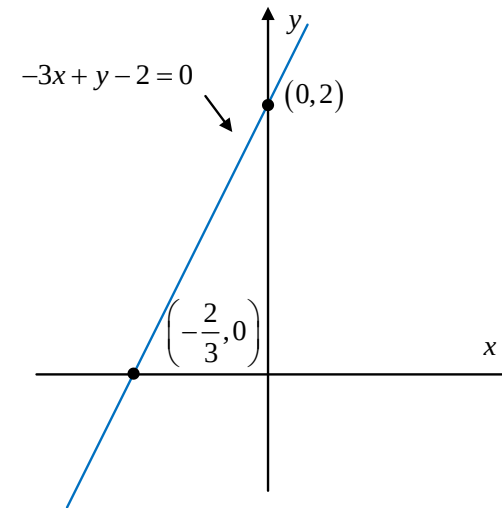
Example 1

$$\text{Minimize } f(x, y) = x^2 + y^2$$

$$\text{Subject to } g(x, y) = -3x + y - 2 = 0$$



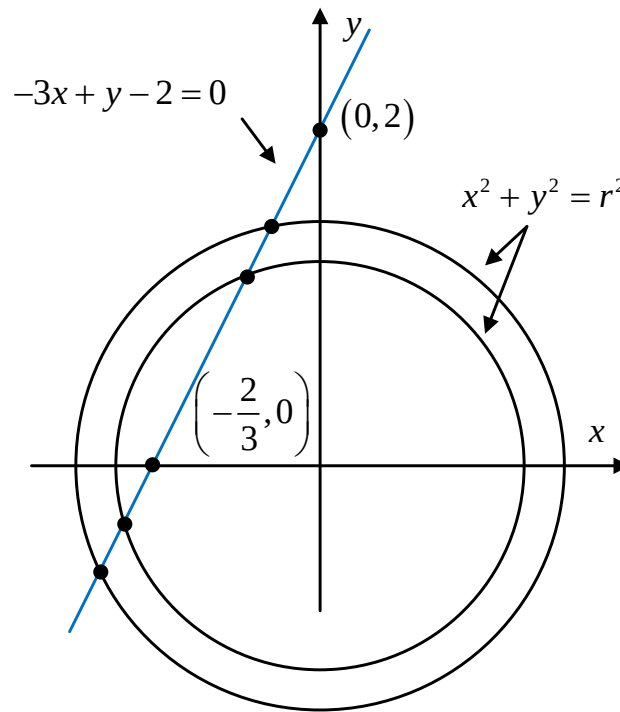
Circles with different sizes



A fixed line

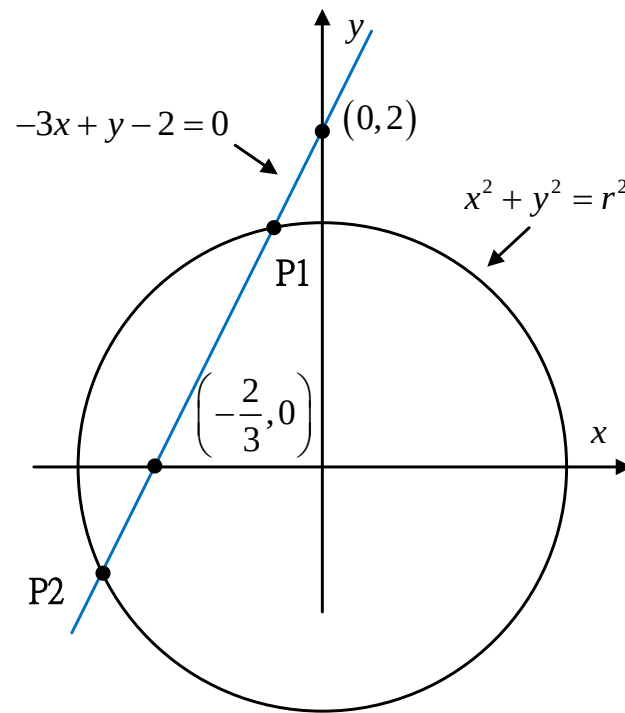
- $x^2 + y^2 = r^2$: circles with different sizes for different r^2
- $-3x + y - 2 = 0$: a fixed line

Example 1



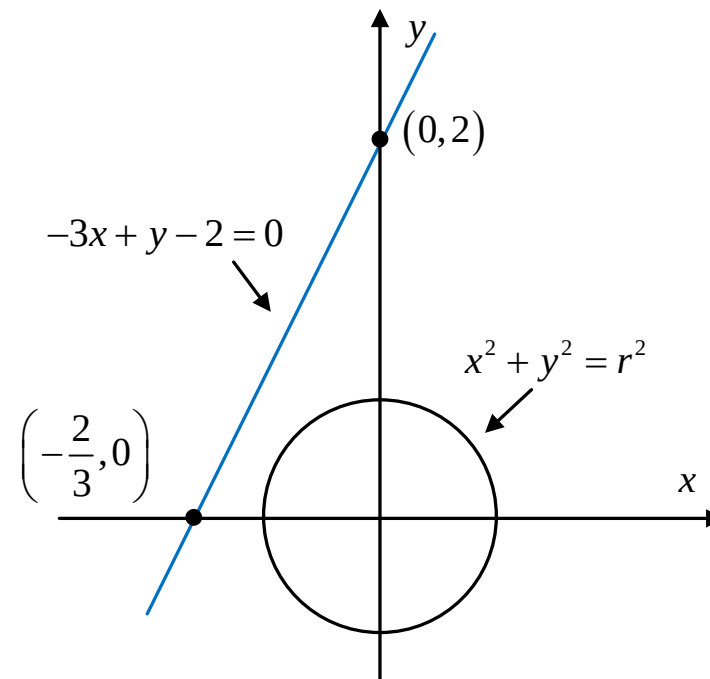
- To find a circle with the smallest r^2 that intersects the line $-3x + y - 2 = 0$
- The intersection point will be the optimal solution

Example 1



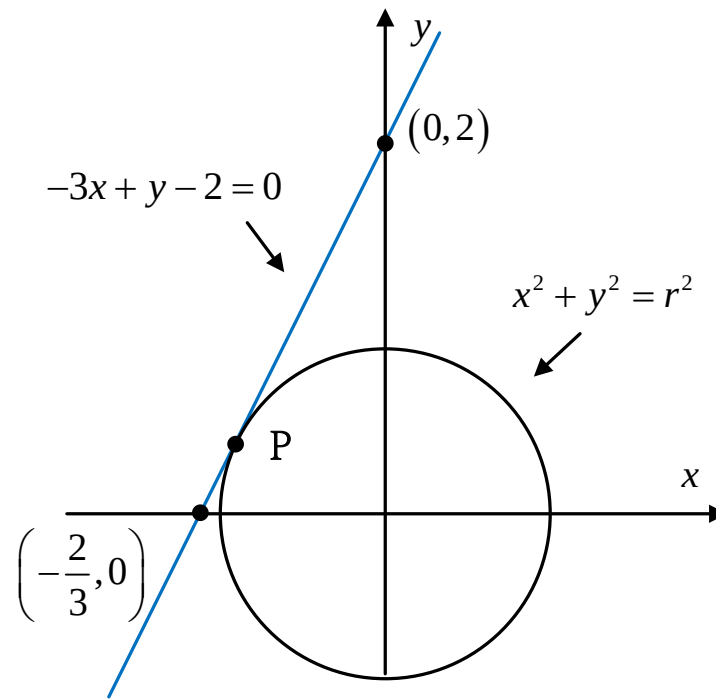
- Two intersection points: when r^2 is too big

Example 1



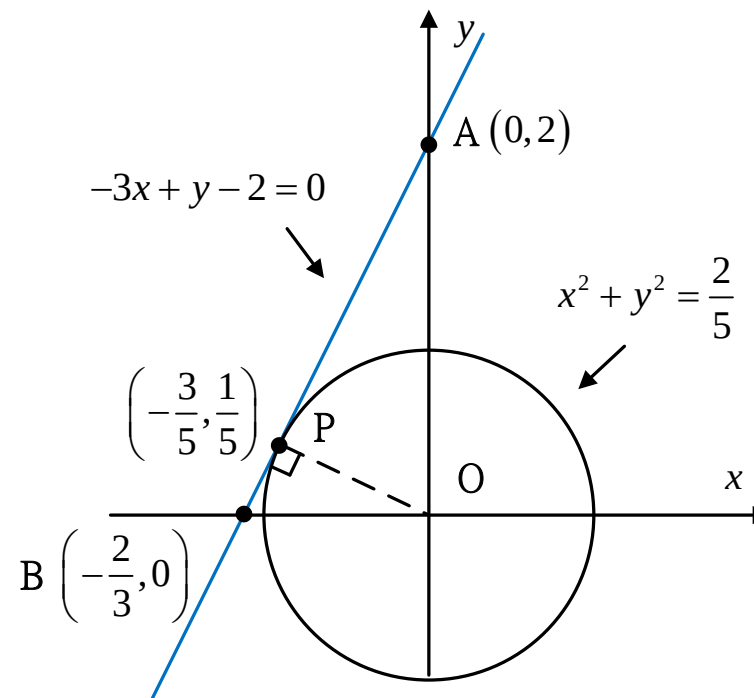
- No intersection point: when r^2 is too small

Example 1



- Only one intersection point: the smallest r^2 with intersection point

Example 1

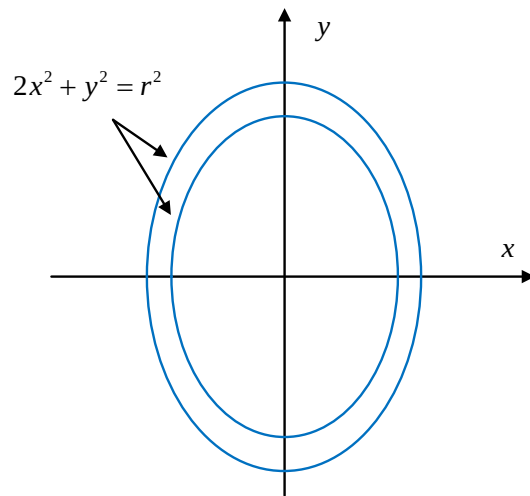


- The intersection point is a tangent point, which can be obtained using high school geometry: $(-0.6, 0.2)$, $f_{\min}(x, y) = 0.4$
- In the triangle OAB , OP is perpendicular to AB (need more details?)

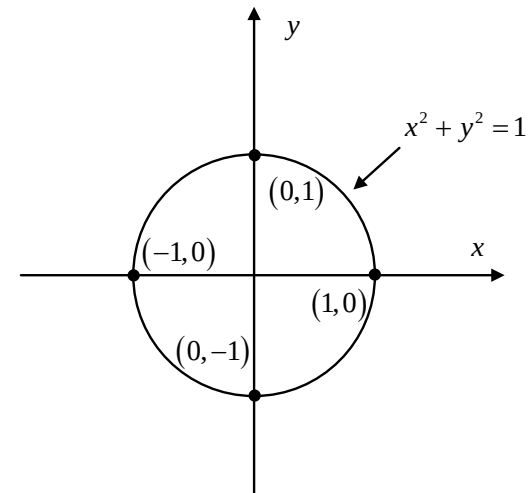
Example 2

Minimize $f(x, y) = 2x^2 + y^2$

Subject to $g(x, y) = x^2 + y^2 - 1 = 0$



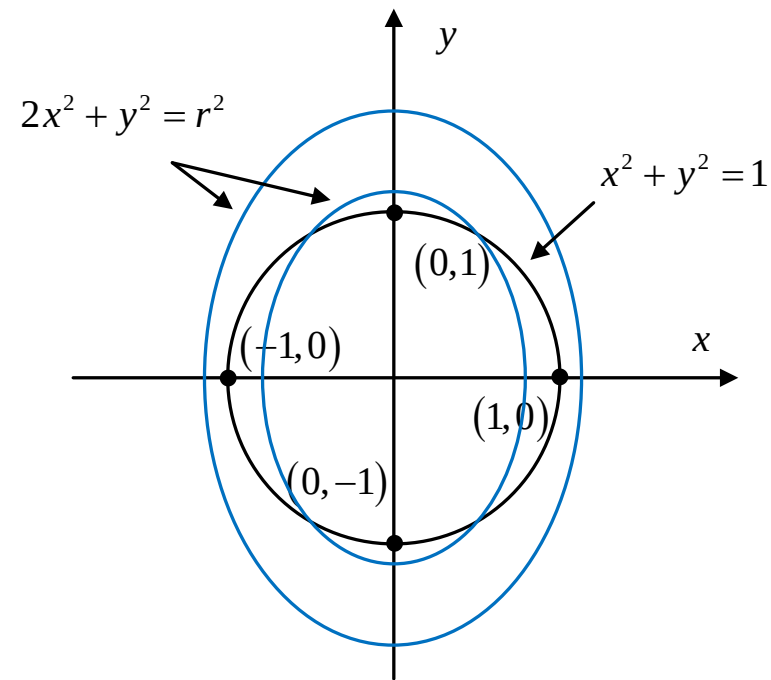
Ellipses with different sizes



A fixed circle

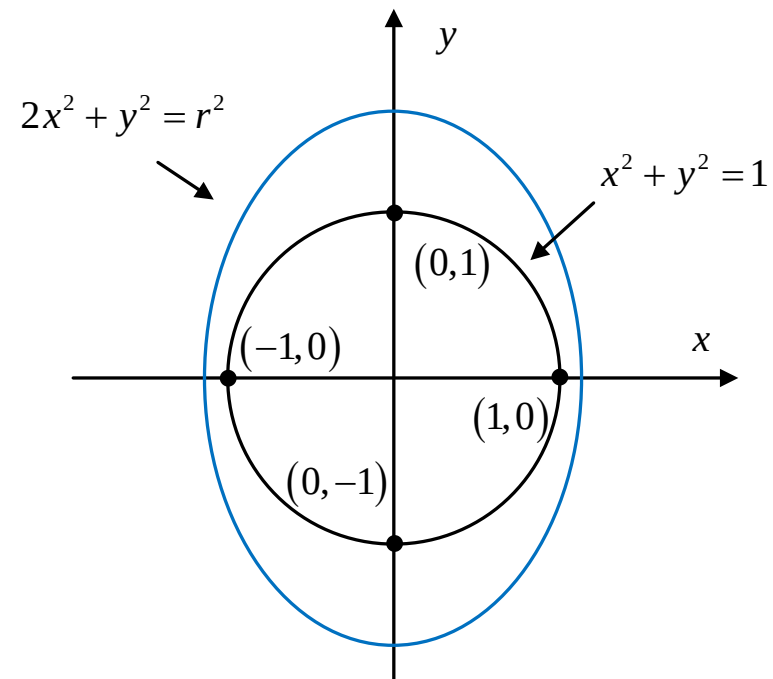
- $2x^2 + y^2 = r^2$: ellipses with different sizes for different r^2
- $x^2 + y^2 - 1 = 0$: a fixed circle

Example 2



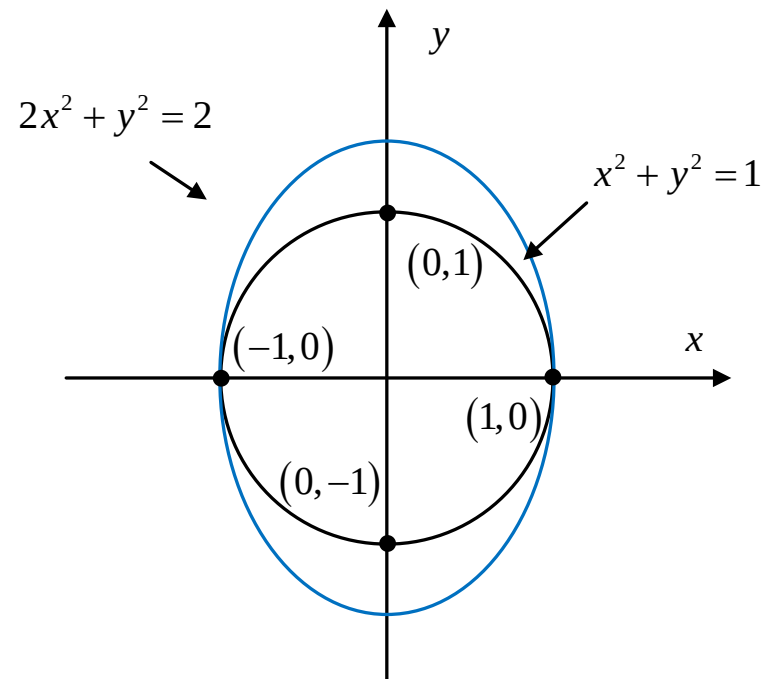
- To find an ellipse with the smallest r^2 that intersects the circle $x^2 + y^2 - 1 = 0$
- The intersection point will be the optimal solution

Example 2



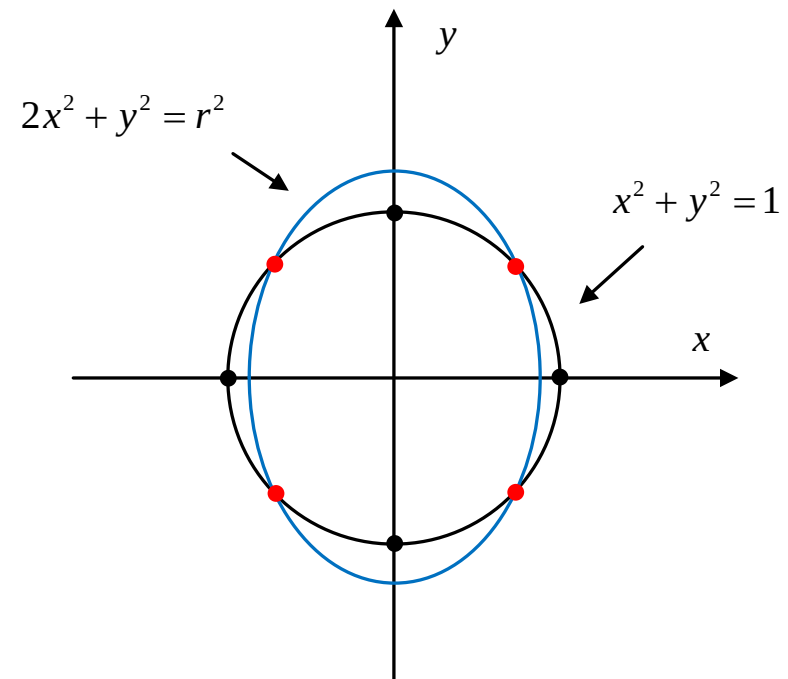
- No intersection point: when r^2 is too big

Example 2



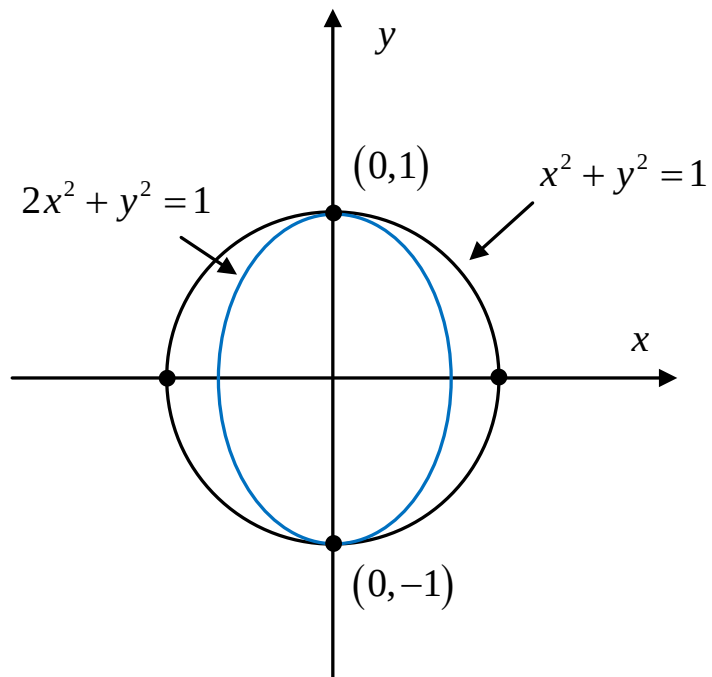
- As r^2 gets smaller, two intersection points

Example 2



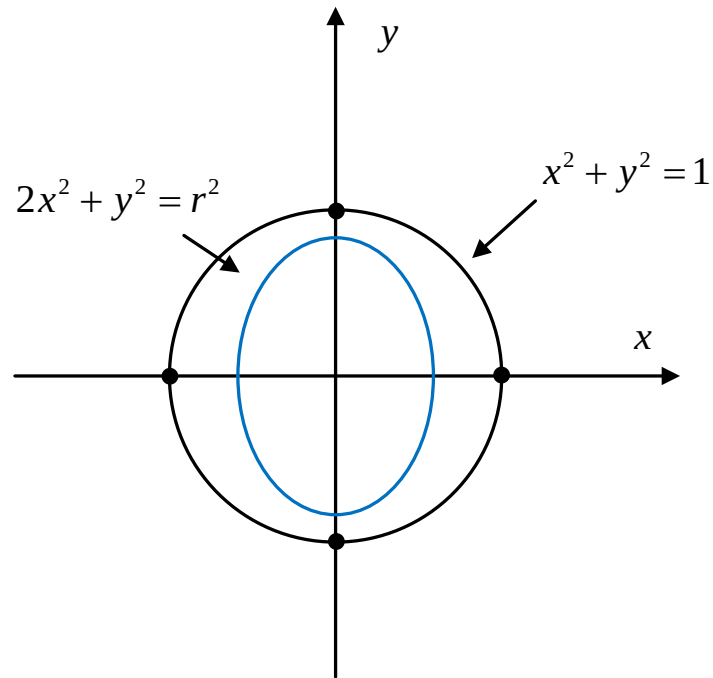
- And smaller: four intersection points

Example 2



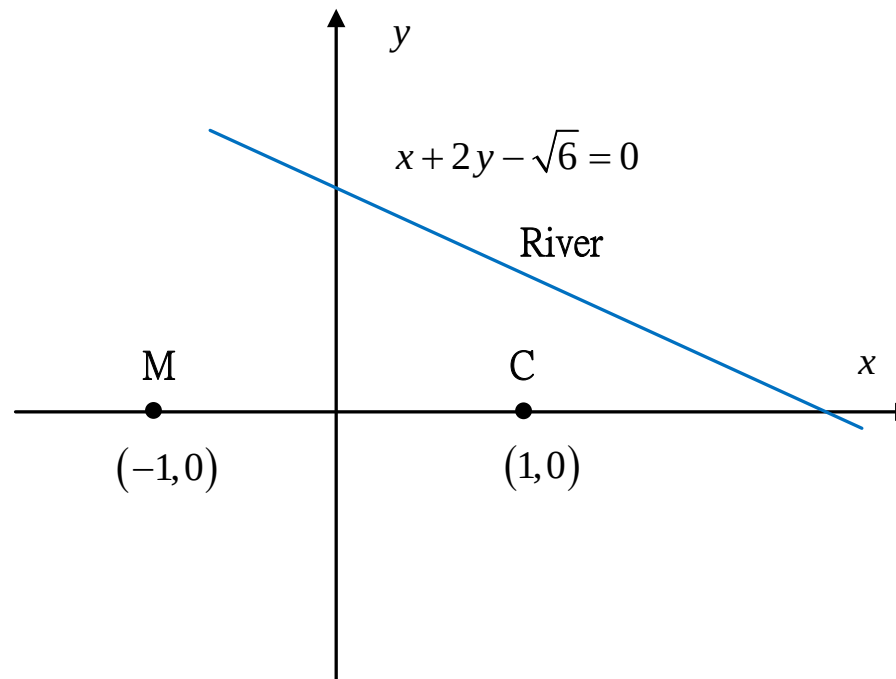
- Two intersection points: $(0, -1)$ and $(0, 1)$, $f_{\min}(x, y) = 2x^2 + y^2 = 1$
- The smallest r^2 with intersection

Example 2



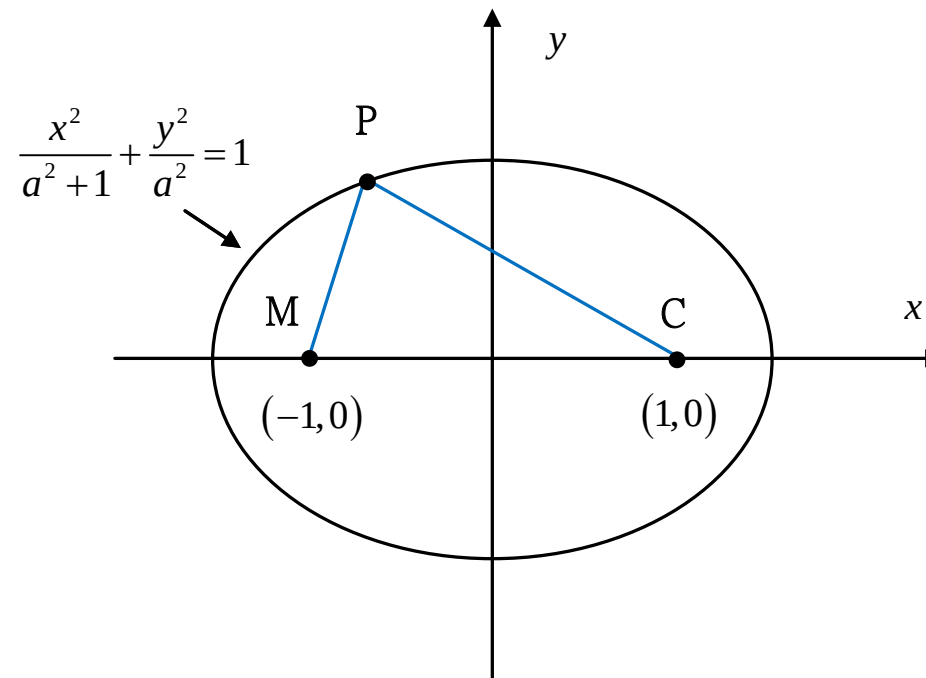
- No intersection point: when r^2 is too small

Milkmaid Problem



- The milkmaid is at point M , the cow is at point C

Milkmaid Problem

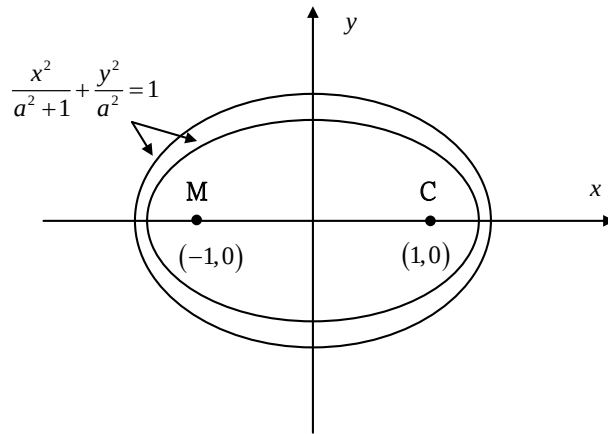


- The two foci of the ellipse $\frac{x^2}{a^2+1} + \frac{y^2}{a^2} = 1$ are M and C
- $MP + PC = 2\sqrt{a^2 + 1}$

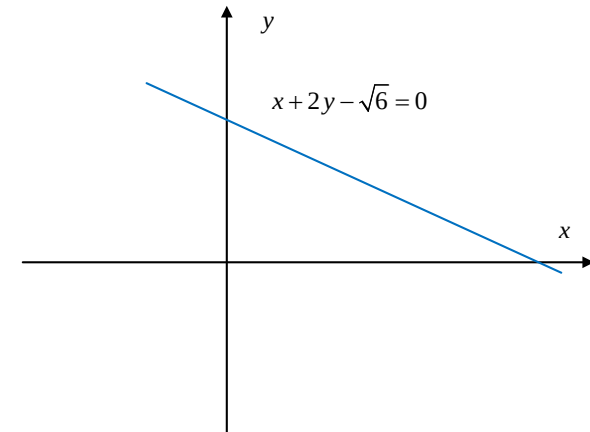
Milkmaid Problem

$$\text{Minimize } f(x, y) = MP + PC = 2\sqrt{a^2 + 1}$$

$$\text{Subject to } g(x, y) = x + 2y - \sqrt{6} = 0$$



Ellipses with different sizes

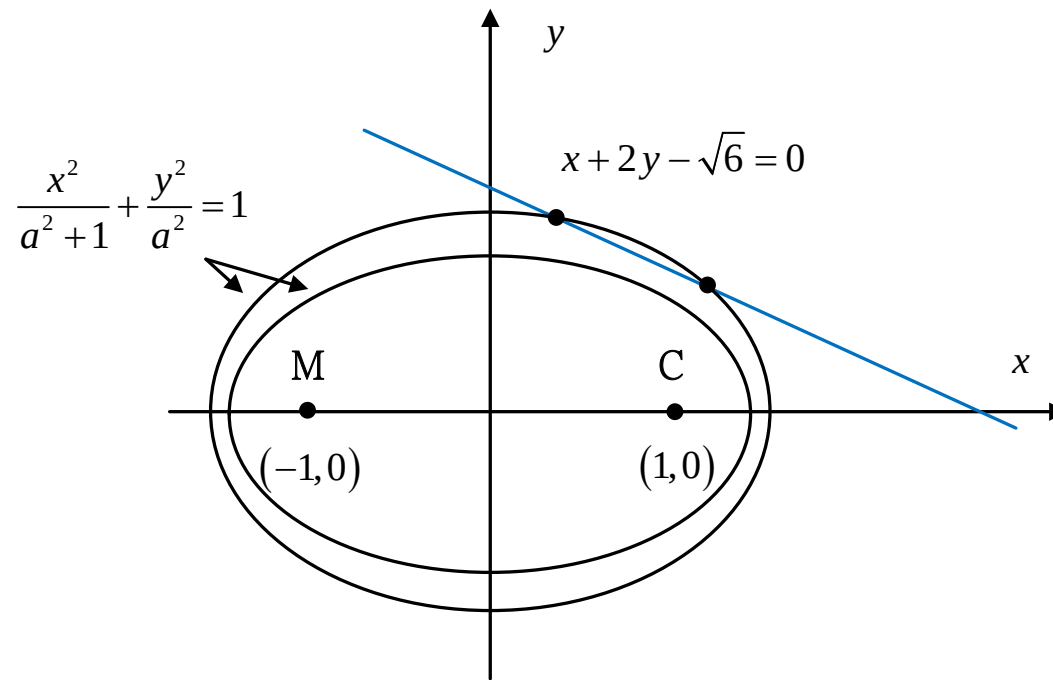


A fixed line

■ $\frac{x^2}{a^2+1} + \frac{y^2}{a^2} = 1$: Ellipses with different sizes for different a^2

■ $x + 2y - \sqrt{6} = 0$: a fixed line

Milkmaid Problem

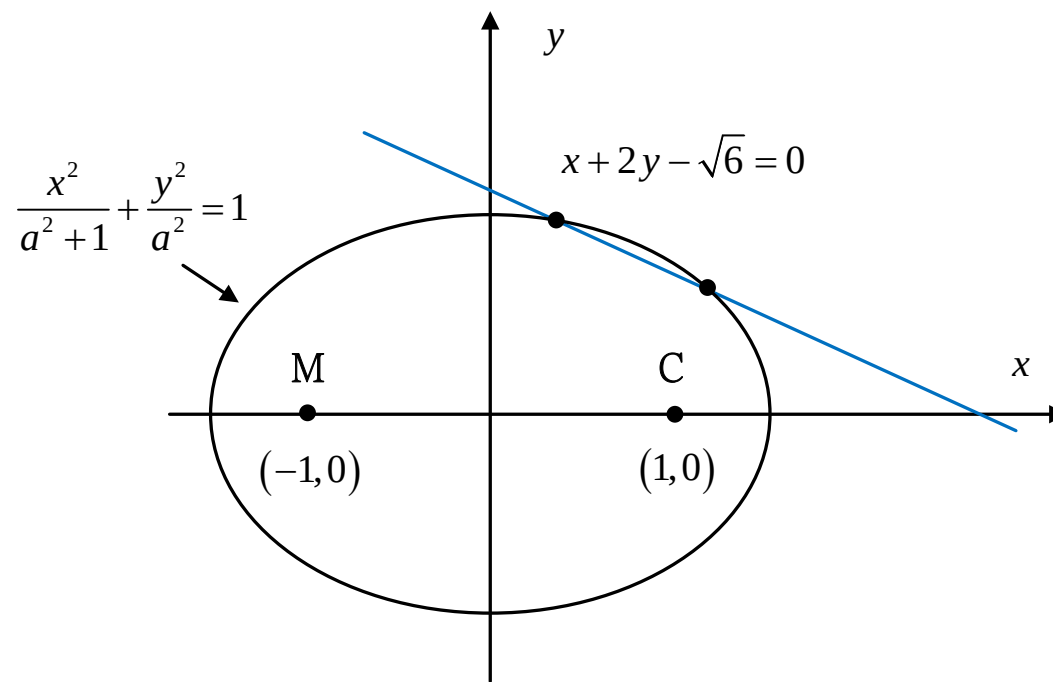


- To find an ellipse with the smallest a^2 that intersects the line $x + 2y - \sqrt{6} = 0$
- The intersection point will be the optimal solution

Milkmaid Problem

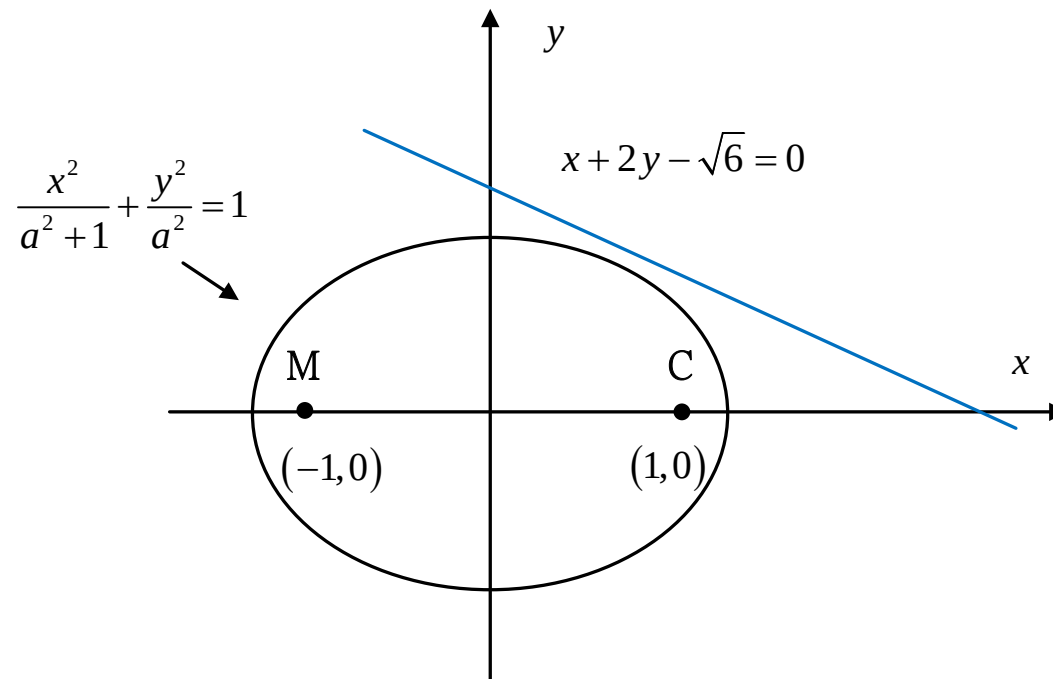
$$\text{Minimize } f(x, y) = MP + PC = 2\sqrt{a^2 + 1}$$

$$\text{Subject to } g(x, y) = x + 2y - \sqrt{6} = 0$$



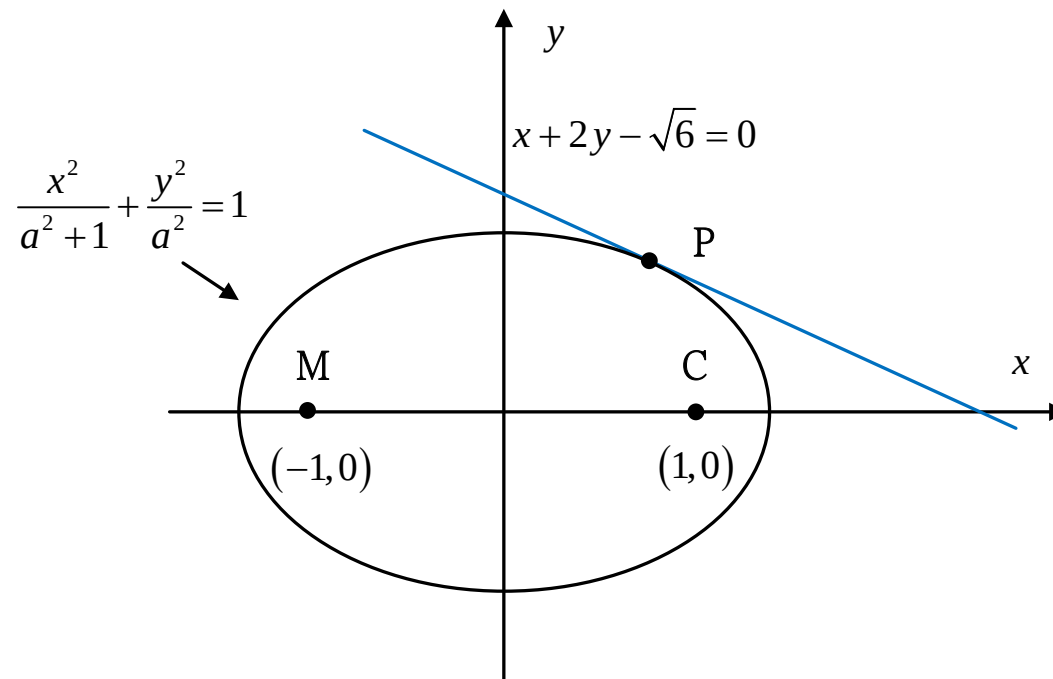
- Two intersection points: when a^2 is too big

Milkmaid Problem



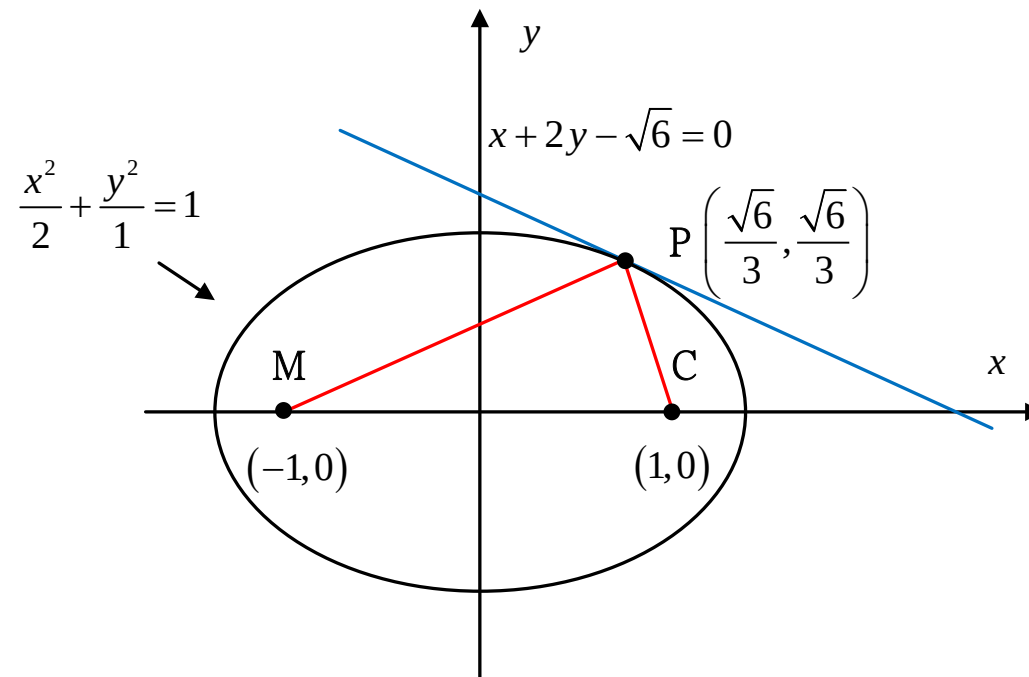
- No intersection point: when a^2 is too small

Milkmaid Problem



- Only one intersection point: the smallest a^2 with intersection point

Milkmaid Problem



- The intersection point is a tangent point, which can be obtained using high school geometry: $\left(\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3}\right)$
- $a^2 = 1, f_{\min}(x, y) = 2\sqrt{2}$

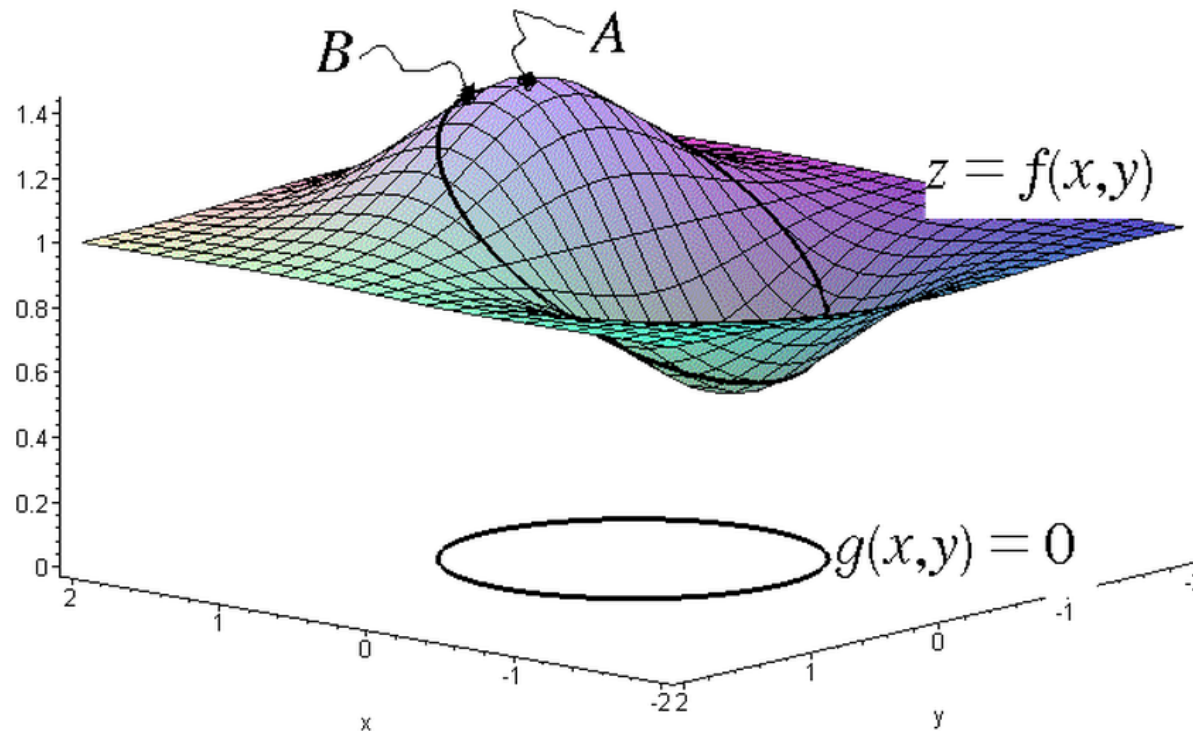
Lagrange Multipliers: Two Variables

Optimize $f(x, y)$

Subject to $g(x, y) = 0$

- Constrained optimization
- Equality constraint
- Three unknown variables x, y, λ and three equations
- Equation 1: $\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x, y)}{\Delta x} = \lambda \lim_{\Delta x \rightarrow 0} \frac{\Delta g(x, y)}{\Delta x}$
- Equation 2: $\lim_{\Delta y \rightarrow 0} \frac{\Delta f(x, y)}{\Delta y} = \lambda \lim_{\Delta y \rightarrow 0} \frac{\Delta g(x, y)}{\Delta y}$
- Equation 3: $g(x, y) = 0$

Lagrange Multipliers



- A : optimum without constraint
- B : optimum with constraint

Lagrange Multipliers: Three Variables

Optimize $f(x, y, z)$

Subject to $g(x, y, z) = 0$

- Can be of higher dimension
- Three unknown variables x, y, z, λ and three equations
- Equation 1: $\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x, y, z)}{\Delta x} = \lambda \lim_{\Delta x \rightarrow 0} \frac{\Delta g(x, y, z)}{\Delta x}$
- Equation 2: $\lim_{\Delta y \rightarrow 0} \frac{\Delta f(x, y, z)}{\Delta y} = \lambda \lim_{\Delta y \rightarrow 0} \frac{\Delta g(x, y, z)}{\Delta y}$
- Equation 3: $\lim_{\Delta z \rightarrow 0} \frac{\Delta f(x, y, z)}{\Delta z} = \lambda \lim_{\Delta z \rightarrow 0} \frac{\Delta g(x, y, z)}{\Delta z}$
- Equation 4: $g(x, y, z) = 0$

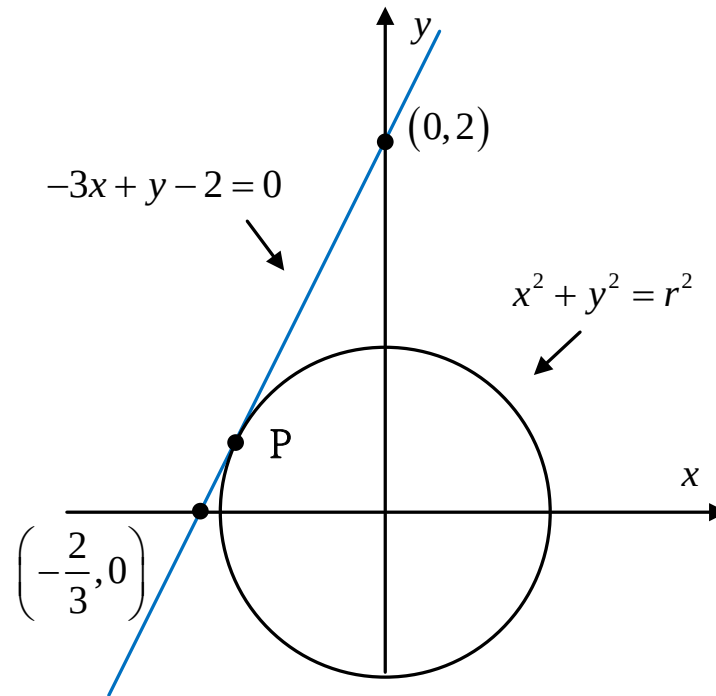
Example 1

Minimize $f(x, y) = x^2 + y^2$

Subject to $g(x, y) = -3x + y - 2 = 0$

- $\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x, y)}{\Delta x} = \lambda \lim_{\Delta x \rightarrow 0} \frac{\Delta g(x, y)}{\Delta x}: 2x = -3\lambda$
- $\lim_{\Delta y \rightarrow 0} \frac{\Delta f(x, y)}{\Delta y} = \lambda \lim_{\Delta y \rightarrow 0} \frac{\Delta g(x, y)}{\Delta y}: 2y = \lambda$
- $-3x + y - 2 = 0$
- Solution: $(-0.6, 0.2), f_{\min}(x, y) = 0.4$

Example 1



- Only one minimum

Example 2

Minimize $f(x, y) = 2x^2 + y^2$

Subject to $g(x, y) = x^2 + y^2 - 1 = 0$

■ $\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x, y)}{\Delta x} = \lambda \lim_{\Delta x \rightarrow 0} \frac{\Delta g(x, y)}{\Delta x}: 4x = 2\lambda x$

■ $\lim_{\Delta y \rightarrow 0} \frac{\Delta f(x, y)}{\Delta y} = \lambda \lim_{\Delta y \rightarrow 0} \frac{\Delta g(x, y)}{\Delta y}: 2y = 2\lambda y$

■ $x^2 + y^2 - 1 = 0$

■ Four solutions:

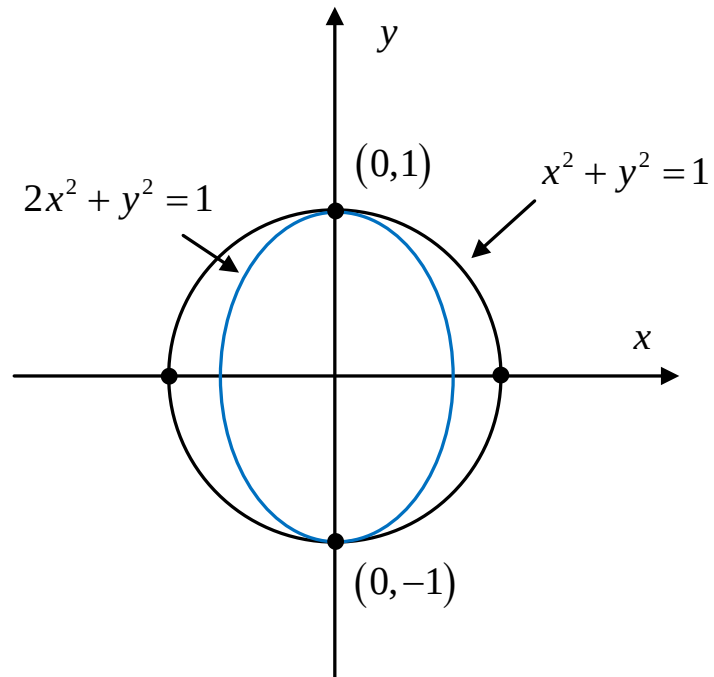
$$(1, 0), f(x, y) = 2$$

$$(-1, 0), f(x, y) = 2$$

$$(0, 1), f(x, y) = 1$$

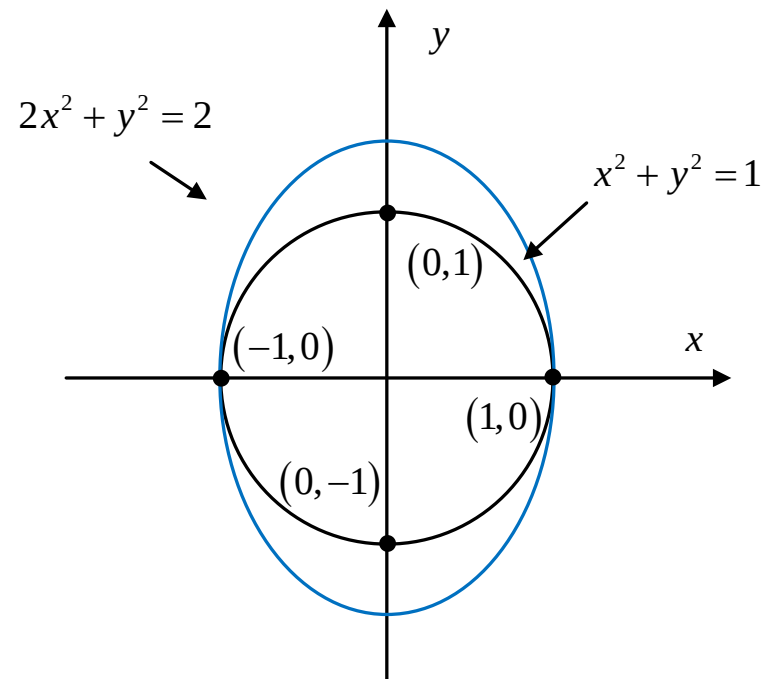
$$(0, -1), f(x, y) = 1$$

Example 2



- $(0, -1)$ and $(0, 1)$, $f_{\min}(x, y) = 2x^2 + y^2 = 1$
- Global minimum

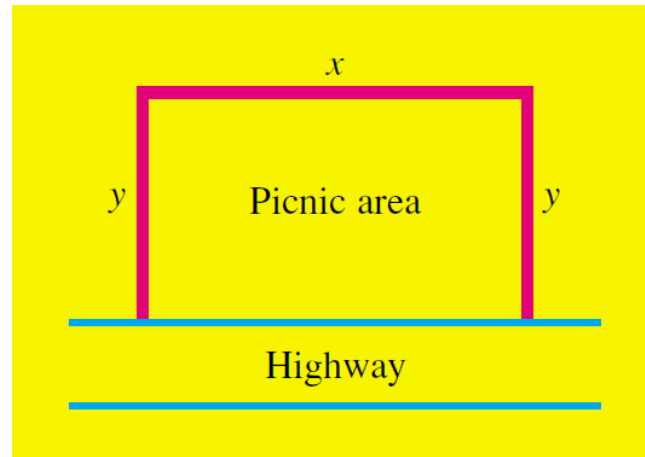
Example 2



■ $(-1, 0)$ and $(1, 0)$, $f(x, y) = 2x^2 + y^2 = 2$

■ Local minimum

Example 3: Rectangular Picnic Area



- The highway department is planning to build a picnic area for motorists along a major highway
- Rectangular with an area of 5,000 square yards ($xy = 5000$), to be fenced off on the three sides not adjacent to the highway
- What is the least amount of fencing ($x + 2y$) that will be needed to complete the job?

Example 3: Rectangular Picnic Area

Minimize $f(x, y) = x + 2y$

Subject to $g(x, y) = xy - 5000 = 0$

■ $\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x, y)}{\Delta x} = \lambda \lim_{\Delta x \rightarrow 0} \frac{\Delta g(x, y)}{\Delta x}: 1 = \lambda y$

■ $\lim_{\Delta y \rightarrow 0} \frac{\Delta f(x, y)}{\Delta y} = \lambda \lim_{\Delta y \rightarrow 0} \frac{\Delta g(x, y)}{\Delta y}: 2 = \lambda x$

■ $xy - 5000 = 0$

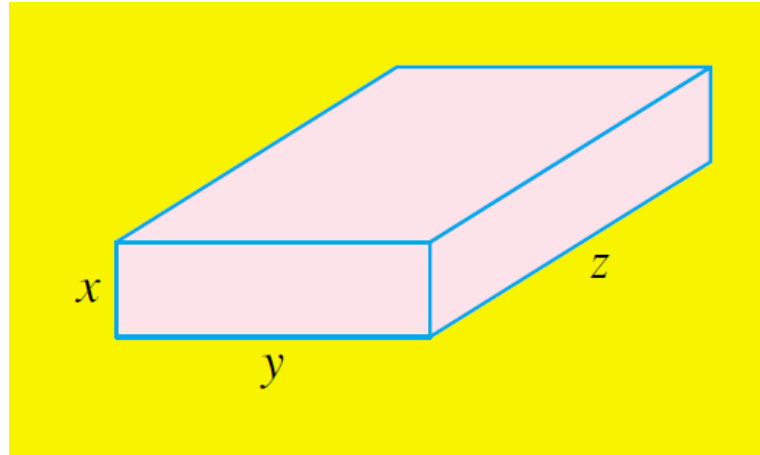
■ Two solutions:

$$(100, 50), f(x, y) = 200$$

$$(-100, -50), f(x, y) = -200$$

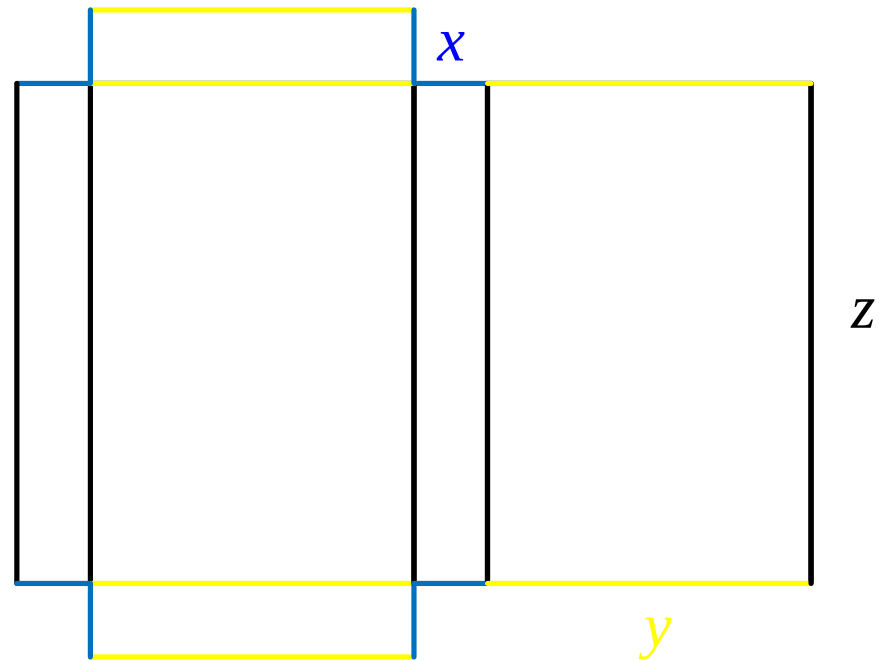
■ The second solution is not reasonable

Example 4: Jewel Box



- A jewel box is to be constructed
- 1 HKD per square inch for the bottom
- 2 HKD per square inch for the sides
- 5 HKD per square inch for the top
- The total volume is 96 inch^3
- What dimensions will minimize the total cost of construction?

Example 4: Jewel Box



- The jewel box unfolded

Example 4: Jewel Box

- Let the box be x inches deep, y inches long, and z inches wide
- Volume of the box: $V = xyz$
- Bottom cost: $1yz$
- Sides cost: $2(2xy + 2xz)$
- Top cost: $5yz$
- Total cost of construction: $C = 1yz + 2(2xy + 2xz) + 5yz = 6yz + 4xy + 4xz$

Example 4: Jewel Box

Minimize $f(x, y, z) = C = 6yz + 4xy + 4xz$

Subject to $g(x, y, z) = V - 96 = xyz - 96 = 0$

$$\blacksquare \lim_{\Delta x \rightarrow 0} \frac{\Delta f(x, y, z)}{\Delta x} = \lambda \lim_{\Delta x \rightarrow 0} \frac{\Delta g(x, y, z)}{\Delta x}: 4y + 4z = \lambda yz$$

$$\blacksquare \lim_{\Delta y \rightarrow 0} \frac{\Delta f(x, y, z)}{\Delta y} = \lambda \lim_{\Delta y \rightarrow 0} \frac{\Delta g(x, y, z)}{\Delta y}: 6z + 4x = \lambda xz$$

$$\blacksquare \lim_{\Delta z \rightarrow 0} \frac{\Delta f(x, y, z)}{\Delta z} = \lambda \lim_{\Delta z \rightarrow 0} \frac{\Delta g(x, y, z)}{\Delta z}: 6y + 4x = \lambda xy$$

$$\blacksquare xyz - 96 = 0$$

$$\blacksquare \text{Solution: } (6, 4, 4)$$

Example 5: Maximum Production Level

- A manufacturer's production is modeled by the Cobb-Douglas function:

$$f(x, y) = 100x^{\frac{4}{3}}y^{\frac{1}{4}}$$

- x represents the units of labor, y represents the units of capital
- Each labor unit costs 200 HKD and each capital unit costs 250 HKD. The total expenses for labor and capital cannot exceed 50,000 HKD .
- Find the maximum production level

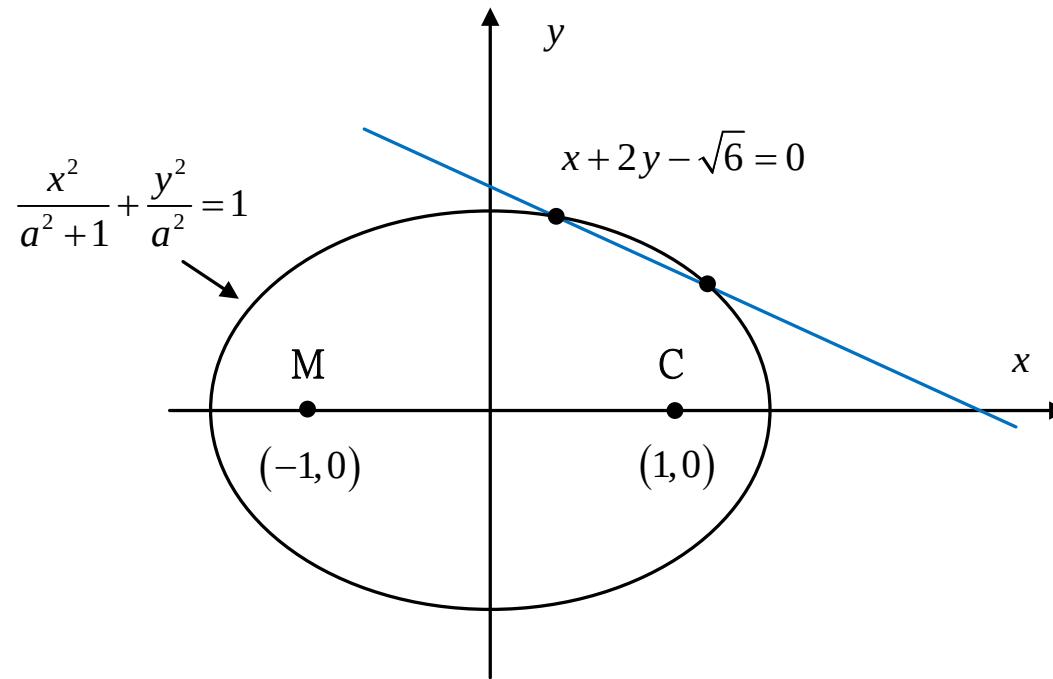
Example 5: Maximum Production Level

Maximize $f(x, y) = 100x^{\frac{3}{4}}y^{\frac{1}{4}}$

Subject to $g(x, y) = 200x + 250y - 50000 = 0$

- $\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x, y)}{\Delta x} = \lambda \lim_{\Delta x \rightarrow 0} \frac{\Delta g(x, y)}{\Delta x} : 75x^{-\frac{1}{4}}y^{\frac{1}{4}} = 200\lambda$
- $\lim_{\Delta y \rightarrow 0} \frac{\Delta f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \lambda \frac{\Delta g(x, y)}{\Delta y} : 25x^{\frac{3}{4}}y^{-\frac{3}{4}} = 250\lambda$
- Used the fact that the derivative of cx^k is ckx^{k-1} , which you will learn in calculus
- $200x + 250y - 50000 = 0$
- Details of the solution not included

Milkmaid Problem



$$\text{Minimize } f(x, y) = 2\sqrt{a^2 + 1} = \sqrt{(x - 1)^2 + y^2} + \sqrt{(x + 1)^2 + y^2}$$

$$\text{Subject to } g(x, y) = x + 2y - \sqrt{6} = 0$$

Milkmaid Problem

- Can still be solved using Lagrange multipliers
- Needs more advanced calculus
- You can come back to this problem in the future after you take the calculus course

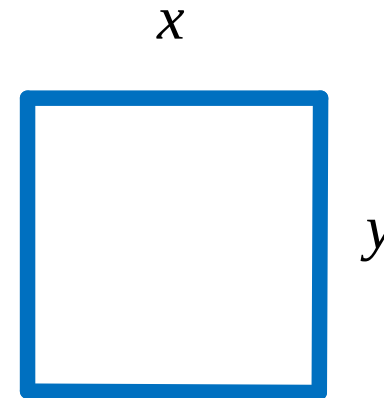
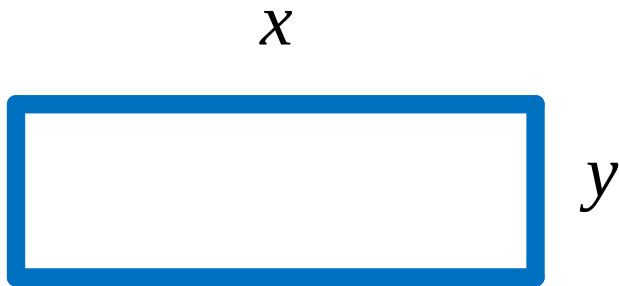
Summary of Lagrange Multipliers

Minimize $f(x, y)$

Subject to $g(x, y) = 0$

- Equality constrained optimization: can be linear and nonlinear
- Visualizing Lagrange multipliers using CalcPlot3D <http://web.monroecc.edu/manila/webfiles/calcNSF/JavaCode/CalcPlot3D.htm>

Exercise 1: Rectangular with a Fixed Perimeter



- Given a rectangular with a fixed perimeter $2(x + y) = c$
- What is the largest area xy ?
- What is your guess?

Exercise 1: Rectangular with a Fixed Perimeter

Minimize $f(x, y) = xy$

Subject to $g(x, y) = 2(x + y) - c = 0$

■ $\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x, y)}{\Delta x} = \lambda \lim_{\Delta x \rightarrow 0} \frac{\Delta g(x, y)}{\Delta x}: y = 2\lambda$

■ $\lim_{\Delta y \rightarrow 0} \frac{\Delta f(x, y)}{\Delta y} = \lambda \lim_{\Delta y \rightarrow 0} \frac{\Delta g(x, y)}{\Delta y}: x = 2\lambda$

■ $2(x + y) - c = 0$

■ Solution: $x = y = \frac{c}{4}$

■ It is a square!

Exercise 1: Rectangular with a Fixed Perimeter

$$\text{Maximize } f(x, y) = xy = x \left(\frac{c}{2} - x \right)$$

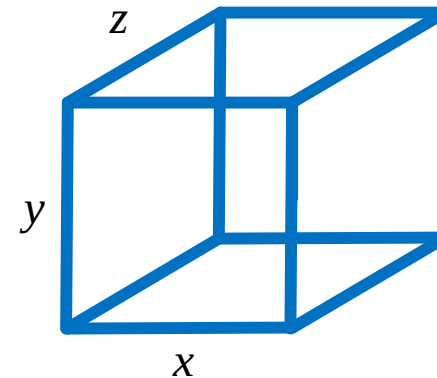
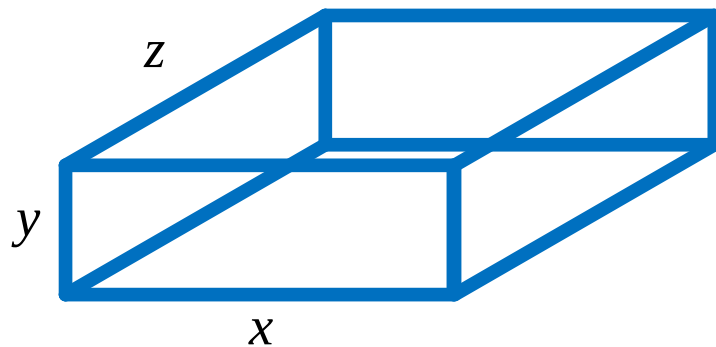
$$\text{Denote } f(x) = x \left(\frac{c}{2} - x \right)$$

- An alternative approach
- $\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x)}{\Delta x} = \frac{c}{2} - 2x$
- $\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x)}{\Delta x} = 0: x = \frac{c}{4}$
- On the left: $\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x)}{\Delta x} < 0$, and increases
- On the right: $\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x)}{\Delta x} > 0$, and increases

Exercise 1: Rectangular with a Fixed Perimeter

- Thus, $x = \frac{c}{4}$ is maximum
- $f(x) = x\left(\frac{c}{2} - x\right)$ is a convex function
- So $x = \frac{c}{4}$ is global maximum
- $y = \frac{c}{4}$
- Not always works!

Exercise 2: Cuboid with a Fixed Surface Area



- Given a cuboid with a fixed surface area $2(xy + yz + zx) = c$
- What is the largest volume xyz ?

Exercise 2: Cuboid with a Fixed Surface Area

Maximize $f(x, y, z) = xyz$

Subject to $g(x, y, z) = 2(xy + yz + zx) - c = 0$

■ $\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x, y, z)}{\Delta x} = \lambda \lim_{\Delta x \rightarrow 0} \frac{\Delta g(x, y, z)}{\Delta x}: yz = 2\lambda y + 2\lambda z$

■ $\lim_{\Delta y \rightarrow 0} \frac{\Delta f(x, y, z)}{\Delta y} = \lambda \lim_{\Delta y \rightarrow 0} \frac{\Delta g(x, y, z)}{\Delta y}: xz = 2\lambda x + 2\lambda z$

■ $\lim_{\Delta z \rightarrow 0} \frac{\Delta f(x, y, z)}{\Delta z} = \lambda \lim_{\Delta z \rightarrow 0} \frac{\Delta g(x, y, z)}{\Delta z}: xy = 2\lambda x + 2\lambda y$

■ $2(xy + yz + zx) - c = 0$

■ Solution: $x = y = z = \sqrt{\frac{c}{6}}$

■ It is a cube!