

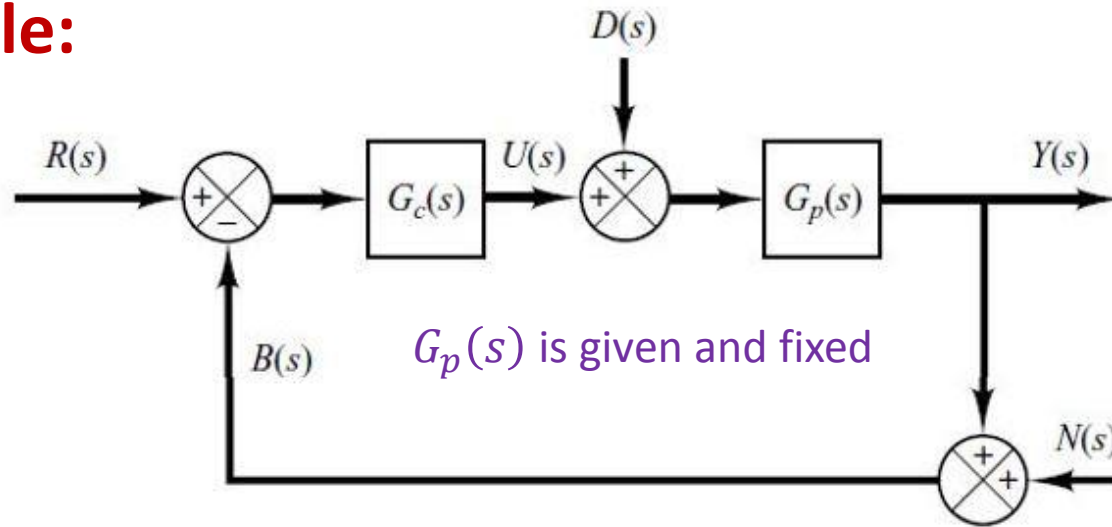
EE3114 Systems and Control

7. Controllers Design

Semester B, 2017-18

1-DoF and 2-DoF PID Controllers

Example:



Closed-loop Transfer Functions:

$$G_{yr}(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

$$G_{yd}(s) = \frac{Y(s)}{D(s)} = \frac{G_p(s)}{1 + G_c(s)G_p(s)}$$

$$G_{yn}(s) = \frac{Y(s)}{N(s)} = \frac{-G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

How many of these are independent?

— Degree of Freedom (DoF)

Closed-loop Transfer Functions:

$$G_{yr}(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

$$G_{yd}(s) = \frac{Y(s)}{D(s)} = \frac{G_p(s)}{1 + G_c(s)G_p(s)}$$

$$G_{yn}(s) = \frac{Y(s)}{N(s)} = \frac{-G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

How many of these are independent?

→ Degree of Freedom (DoF)



$$G_{yr}(s) = G_c(s)G_{yd}(s)$$

$$G_{yn}(s) = -G_c(s)G_{yd}(s)$$

1 DoF --- If one of these is fixed, the other two are linearly dependent

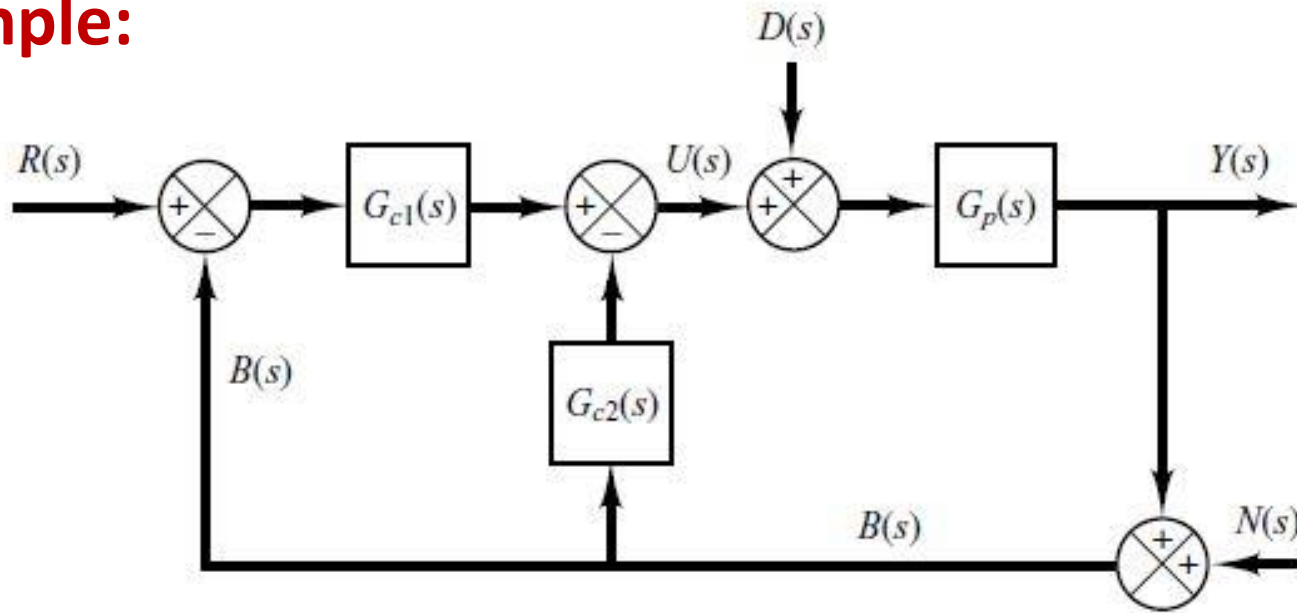
Given $G_{yd}(s) \rightarrow G_{yr}(s) = -G_{yn}(s)$
Linearly dependent



$$G_{yr}(s) = -G_{yn}(s)$$

$G_1(s)$ and $G_2(s)$ are linearly dependent if there are nonzero constants c_1 and c_2 such that $c_1 G_1(s) + c_2 G_2(s) = 0$ for all s

Example:



$$G_{yr}(s) = \frac{Y(s)}{R(s)} = \frac{G_{c1}(s)G_p(s)}{1 + (G_{c1}(s) + G_{c2}(s))G_p(s)}$$

$$G_{yd}(s) = \frac{Y(s)}{D(s)} = \frac{G_p(s)}{1 + (G_{c1}(s) + G_{c2}(s))G_p(s)}$$

$$G_{yn}(s) = \frac{Y(s)}{N(s)} = \frac{(G_{c1}(s) + G_{c2}(s))G_p(s)}{1 + (G_{c1}(s) + G_{c2}(s))G_p(s)}$$

$$G_{yr}(s) = \frac{Y(s)}{R(s)} = \frac{G_{c1}(s)G_p(s)}{1 + (G_{c1}(s) + G_{c2}(s))G_p(s)}$$

$$G_{yd}(s) = \frac{Y(s)}{D(s)} = \frac{G_p(s)}{1 + (G_{c1}(s) + G_{c2}(s))G_p(s)}$$

$$G_{yn}(s) = \frac{Y(s)}{N(s)} = \frac{(G_{c1}(s) + G_{c2}(s))G_p(s)}{1 + (G_{c1}(s) + G_{c2}(s))G_p(s)}$$



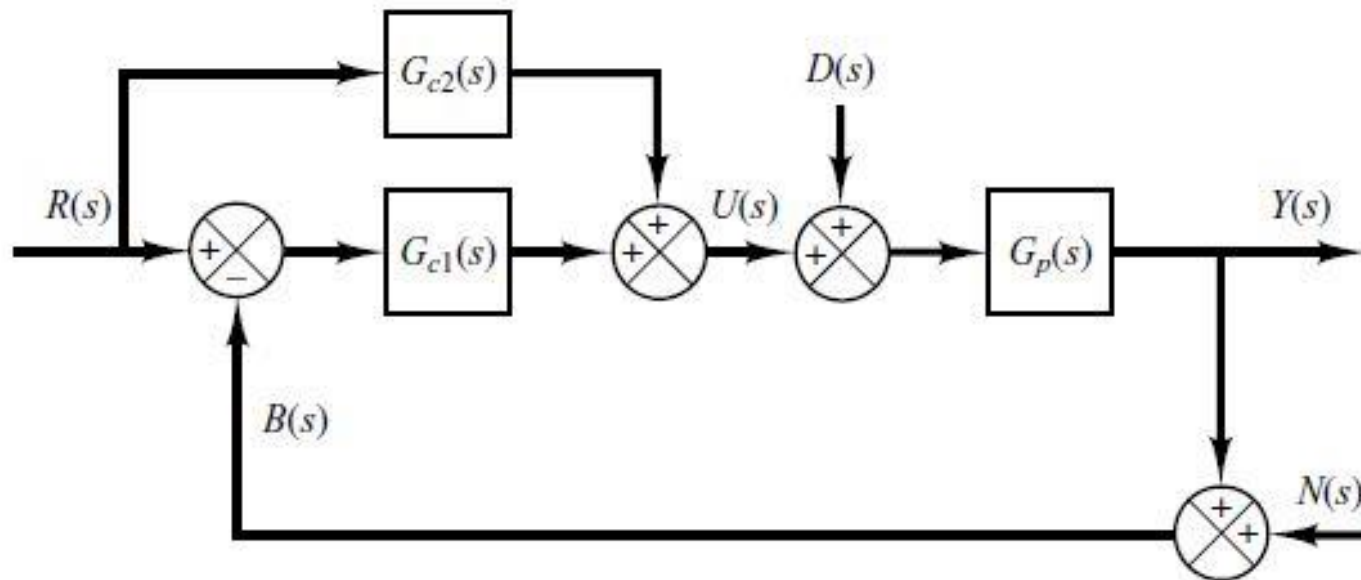
$$G_{yr}(s) = G_{c1}(s)G_{yd}(s)$$

$$G_{yn}(s) = (G_{c1}(s) + G_{c2}(s))G_{yd}$$

2 DoF --- If one of these is fixed,
the other two are linearly
independent

Given $G_{yd}(s) \rightarrow G_{yr}(s)$ and
 $G_{yn}(s)$ are linearly independent

Example:



$$G_{yr} = \frac{Y(s)}{R(s)} = \frac{G_{c1}G_p}{1 + G_{c1}G_p} + \frac{G_{c2}G_p}{1 + G_{c1}G_p}$$

$$G_{yd} = \frac{Y(s)}{D(s)} = \frac{G_p}{1 + G_{c1}G_p}$$

$$G_{yn} = \frac{Y(s)}{N(s)} = -\frac{G_{c1}G_p}{1 + G_{c1}G_p}$$

Determine its DoF ?

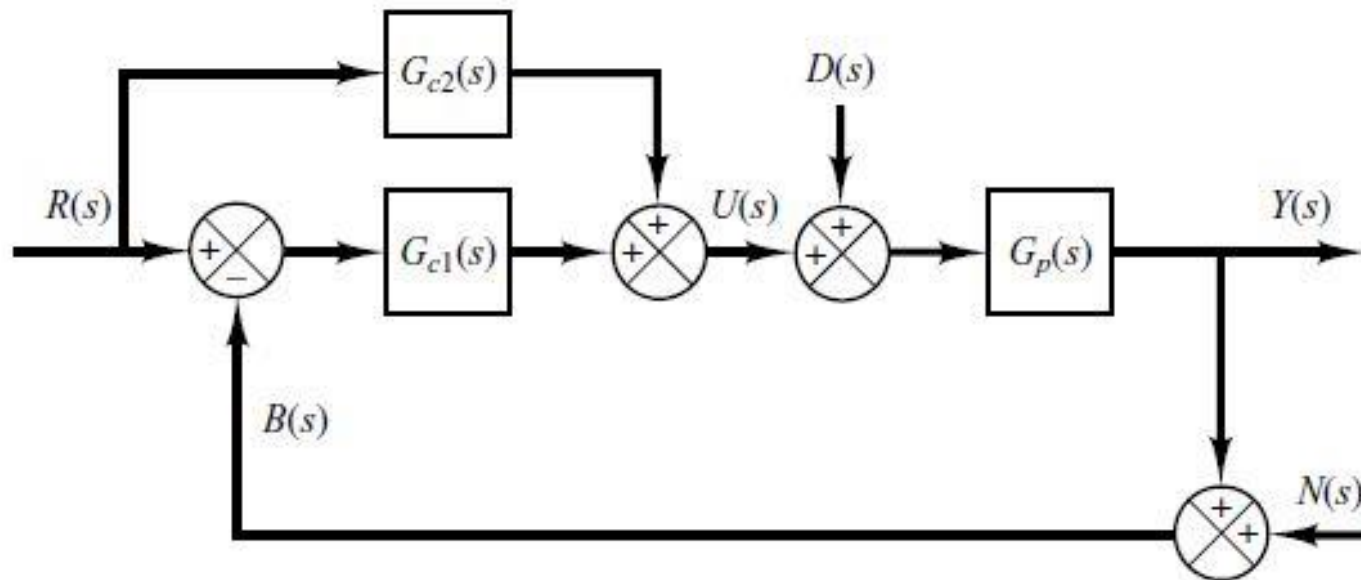
1 DoF ?

2 DoF ?

3 DoF ?

(10 minutes)

Example:



$$G_{yr} = \frac{Y(s)}{R(s)} = \frac{G_{c1}G_p}{1 + G_{c1}G_p} + \frac{G_{c2}G_p}{1 + G_{c1}G_p}$$

$$G_{yd} = \frac{Y(s)}{D(s)} = \frac{G_p}{1 + G_{c1}G_p}$$

$$G_{yn} = \frac{Y(s)}{N(s)} = -\frac{G_{c1}G_p}{1 + G_{c1}G_p}$$

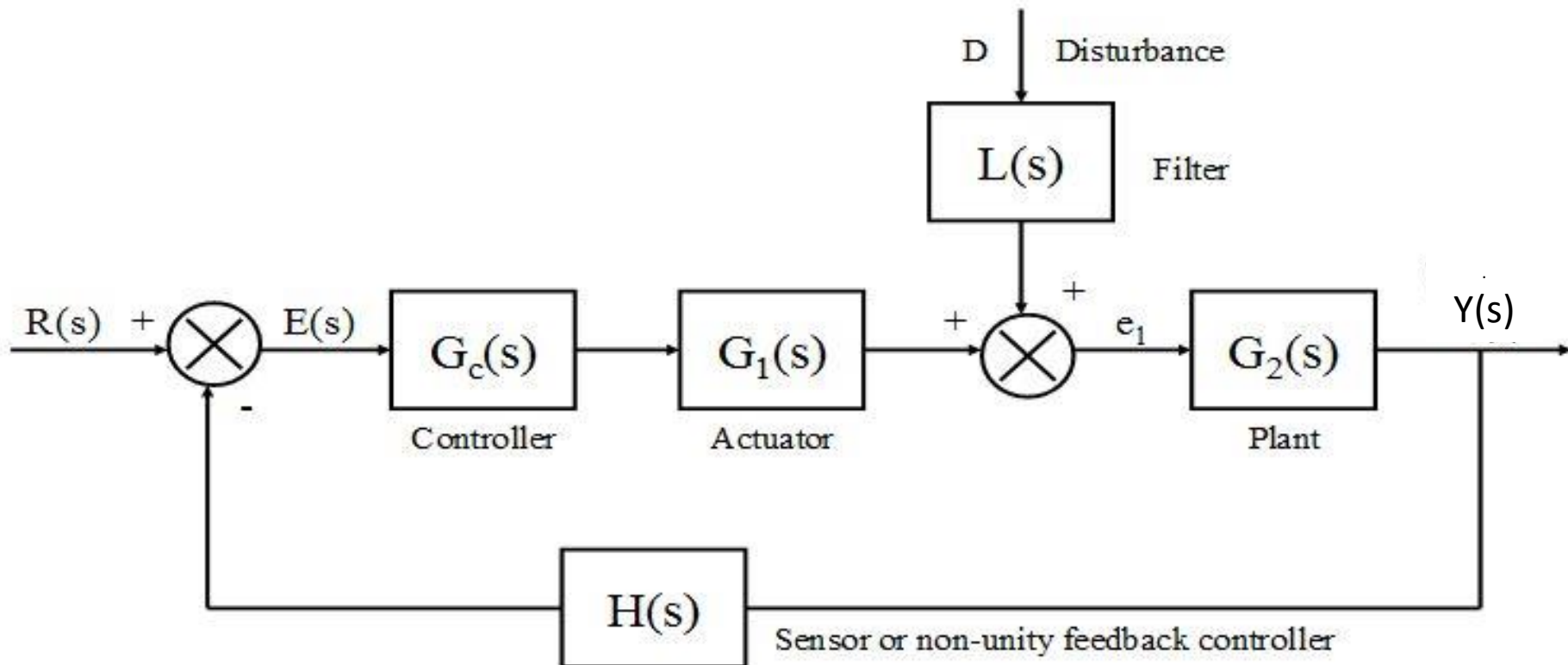
Solution: 2 DoF

$$G_{yr} = G_{c2}G_{yd} + \frac{G_p - G_{yd}}{G_p}$$

$$G_{yn} = \frac{G_{yd} - G_p}{G_p}$$

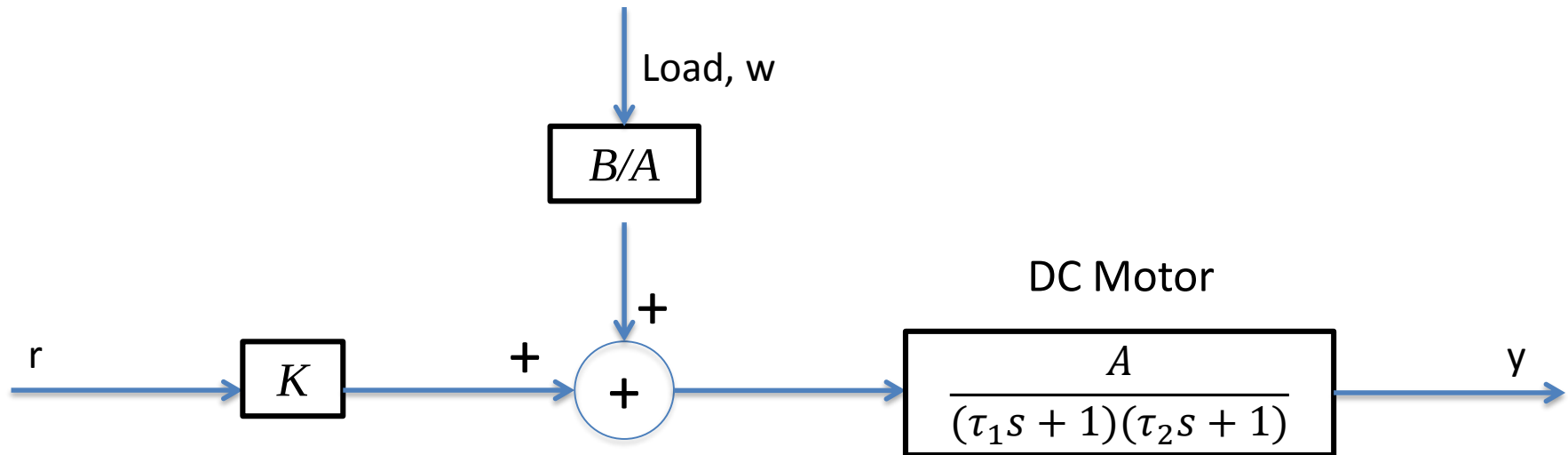
Advantages of Feedback

To study controller performance under **imperfect** conditions:
disturbance, noise, uncertainty, ...



Presence of Disturbance

Example: Load disturbance $w(t)$ on the torque

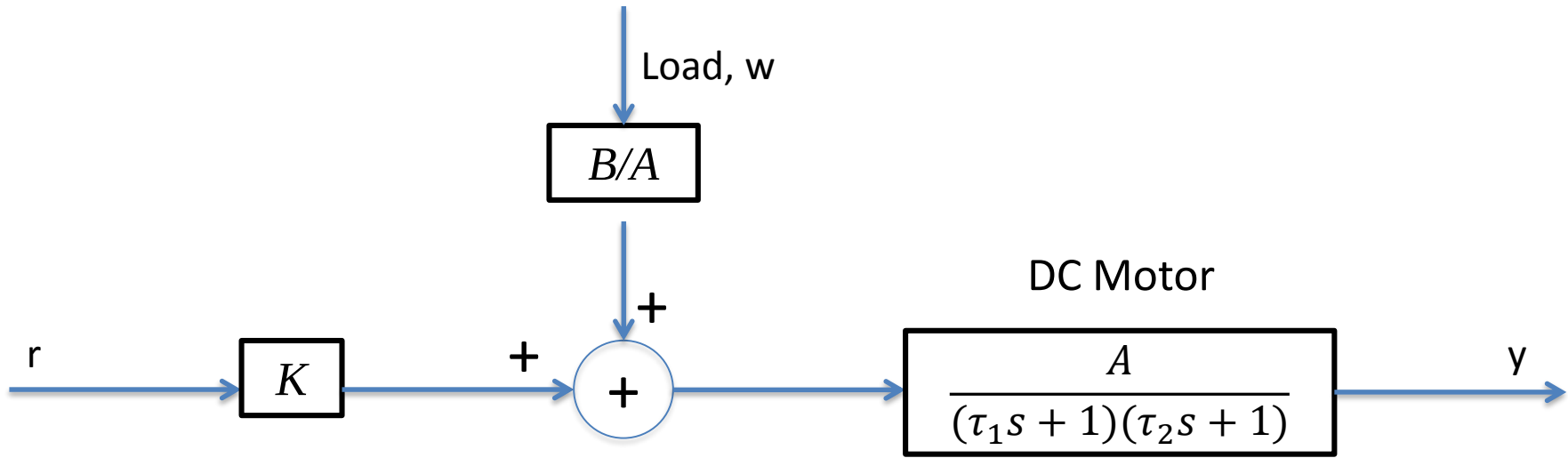


(B/A is inserted as scaling constants; B is to be determined)

Consider the effect of constant disturbance: $w(t) = d \cdot 1(t)$

with $1(t)$ being unit step function $\rightarrow w(s) = \frac{d}{s}$

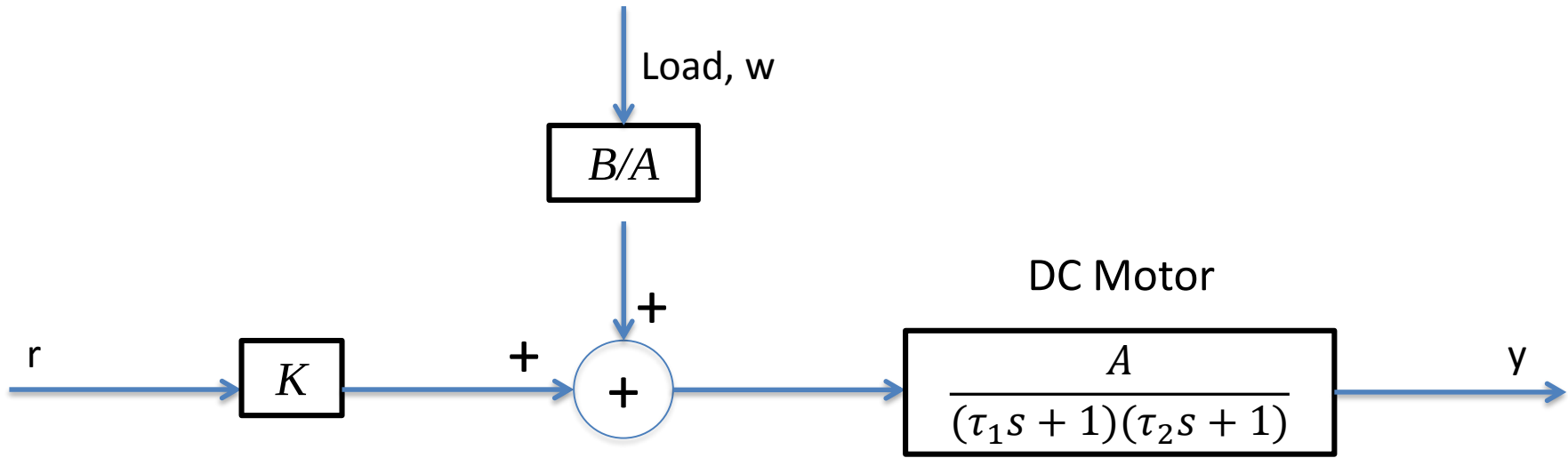
Open-Loop Control



$$Y(s) = \frac{A}{(\tau_1 s + 1)(\tau_2 s + 1)} \left[K_o \frac{r_0}{s} + \frac{B d}{A s} \right]$$

Find: $y(\infty) = ?$ (5 minutes)

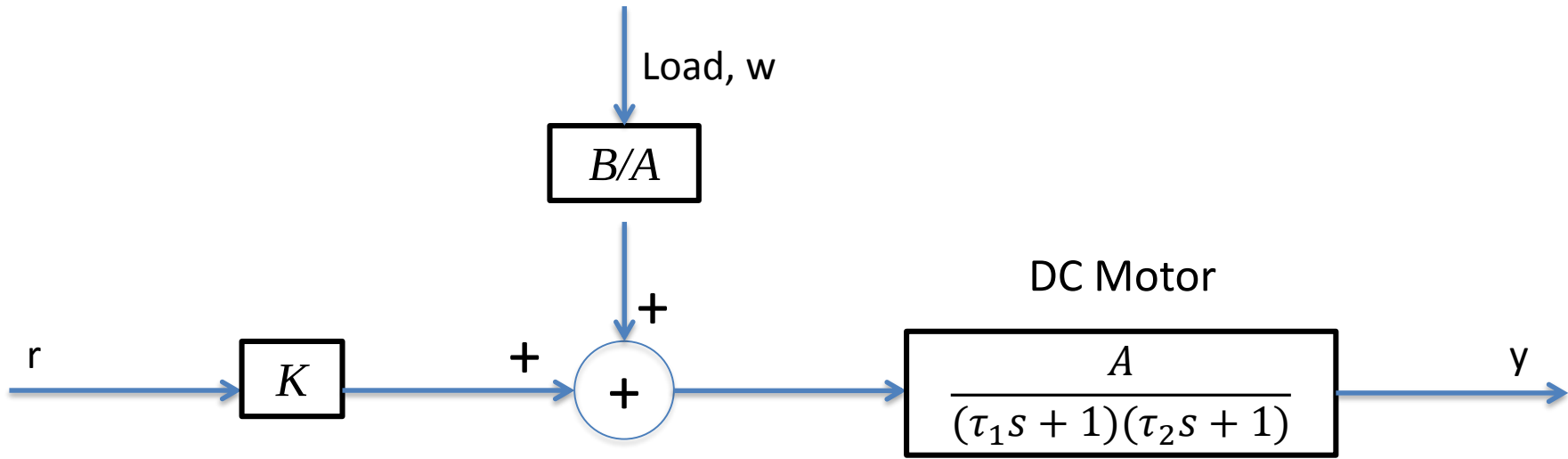
Open-Loop Control



$$Y(s) = \frac{A}{(\tau_1 s + 1)(\tau_2 s + 1)} \left[K_o \frac{r_0}{s} + \frac{B}{A} \frac{d}{s} \right]$$

Solution: $y(\infty) = \lim_{s \rightarrow 0} sY(s) = A \left(K_o r_0 + \frac{B}{A} d \right)$

Open-Loop Control



$$Y(s) = \frac{A}{(\tau_1 s + 1)(\tau_2 s + 1)} \left[K_o \frac{r_0}{s} + \frac{B d}{A s} \right]$$

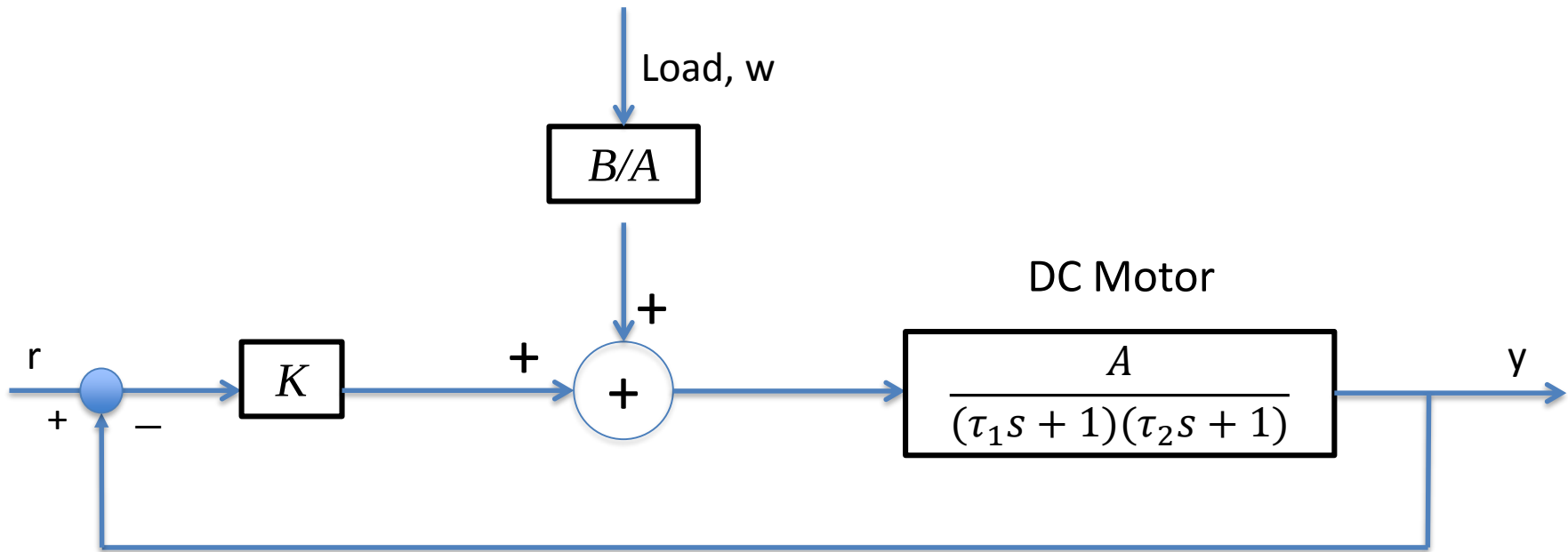
$$y(\infty) = \lim_{s \rightarrow 0} sY(s) = A \left(K_o r_0 + \frac{B}{A} d \right)$$

Open-loop design $\rightarrow K_o = \frac{1}{A} \rightarrow y(\infty) = r_0 + Bd$

$\rightarrow y(\infty)$ deviates from the reference value r_0 by $e_o(\infty) = Bd$

(If no disturbance, $d = 0$, then $e_o(\infty) = 0$)

Closed-Loop Control



$$R(s) = \frac{r_0}{s}, \quad w(t) = \frac{B}{A} d \, 1(t)$$

→
$$Y(s) = \frac{AK_c r_0 + Bd}{s[(\tau_1 s + 1)(\tau_2 s + 1) + AK_c]}$$

Closed-Loop Control

$$Y(s) = \frac{AK_c r_0 + Bd}{s[(\tau_1 s + 1)(\tau_2 s + 1) + AK_c]}$$

- Find: $y(\infty) = ?$ (5 minutes)

Closed-Loop Control

$$Y(s) = \frac{AK_c r_0 + Bd}{s[(\tau_1 s + 1)(\tau_2 s + 1) + AK_c]}$$

Solution:

$$y(\infty) = \lim_{s \rightarrow 0} sY(s) = \frac{AK_c r_0 + Bd}{1 + AK_c}$$

Closed-loop design $\rightarrow AK_c \gg 1 \rightarrow y(\infty) \approx \frac{AK_c r_0 + Bd}{AK_c} = r_0 + \frac{Bd}{AK_c}$

$\rightarrow y(\infty)$ deviates from desired value r_0 by

$$e_c(\infty) \approx \frac{Bd}{AK_c}$$

(If K_c is large, then $e_c(\infty)$ is small)

Comparison

Open-loop: $y(\infty) = r_0 + Bd$

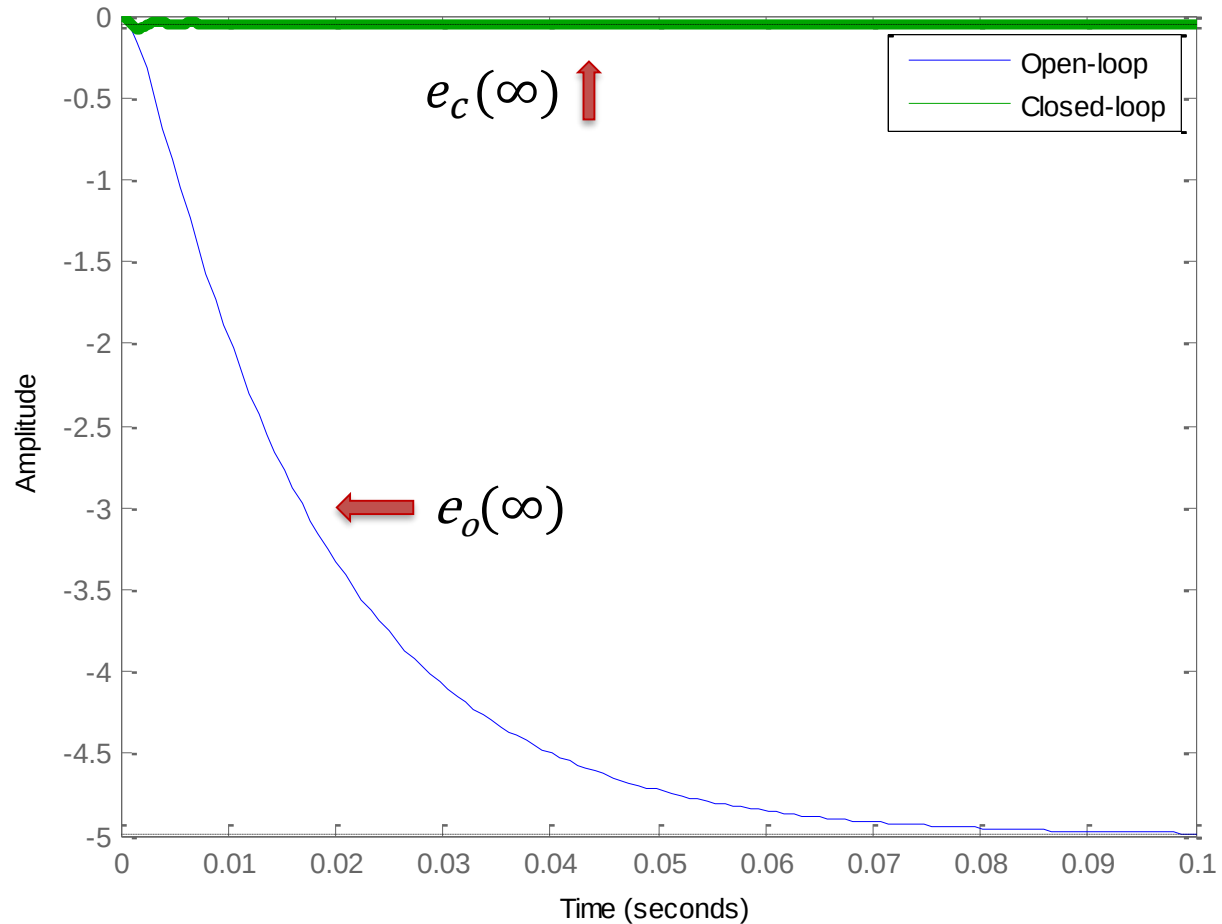
Closed-loop: $y(\infty) = r_0 + \frac{Bd}{AK_c}$



$$e_c(\infty) = \frac{e_o(\infty)}{AK_c} \ll e_o(\infty) \quad (AK_c \gg 1)$$

Comparison

Example:



%MatLab code

```
step(zpk([],[-60 -600],-5*60*600),tf(-180000,[1 660 3600000]))
```

```
legend('Open-loop','Closed-loop')
```

System Parameter Variation

- Model and/or parameters never exactly match reality, e.g.,
 - Insufficient knowledge about the system
 - Fatigue, aging
 - Environmental changes
 - **Example:** Motor gain $A \rightarrow A + \Delta A$
(e.g., a change in operating temperature caused the motor gain to drift from its original value)
- Q:** How does this uncertainty affect the performance?

Open-Loop Control

$$Y(s) = \frac{A+\Delta A}{(\tau_1 s+1)(\tau_2 s+1)} K_o \frac{r_0}{s}$$

- Open-loop design $\rightarrow K_o = \frac{1}{A}$

- Final value theorem $\rightarrow$

$$y(\infty) = \left(1 + \frac{\Delta A}{A}\right) r_0$$

- $y(\infty)$ deviates from reference value r_0 by

$$e_o(\infty) = \frac{\Delta A}{A} r_0$$

- $\frac{e_o(\infty)}{r_0} = \frac{\Delta A}{A}$ called % of error or % of deviation

Closed-Loop Control

$$Y(s) = \frac{(A + \Delta A)K_{cl}r_0}{s[(\tau_1 s + 1)(\tau_2 s + 1) + (A + \Delta A)K_c]}$$

- Final value theorem →

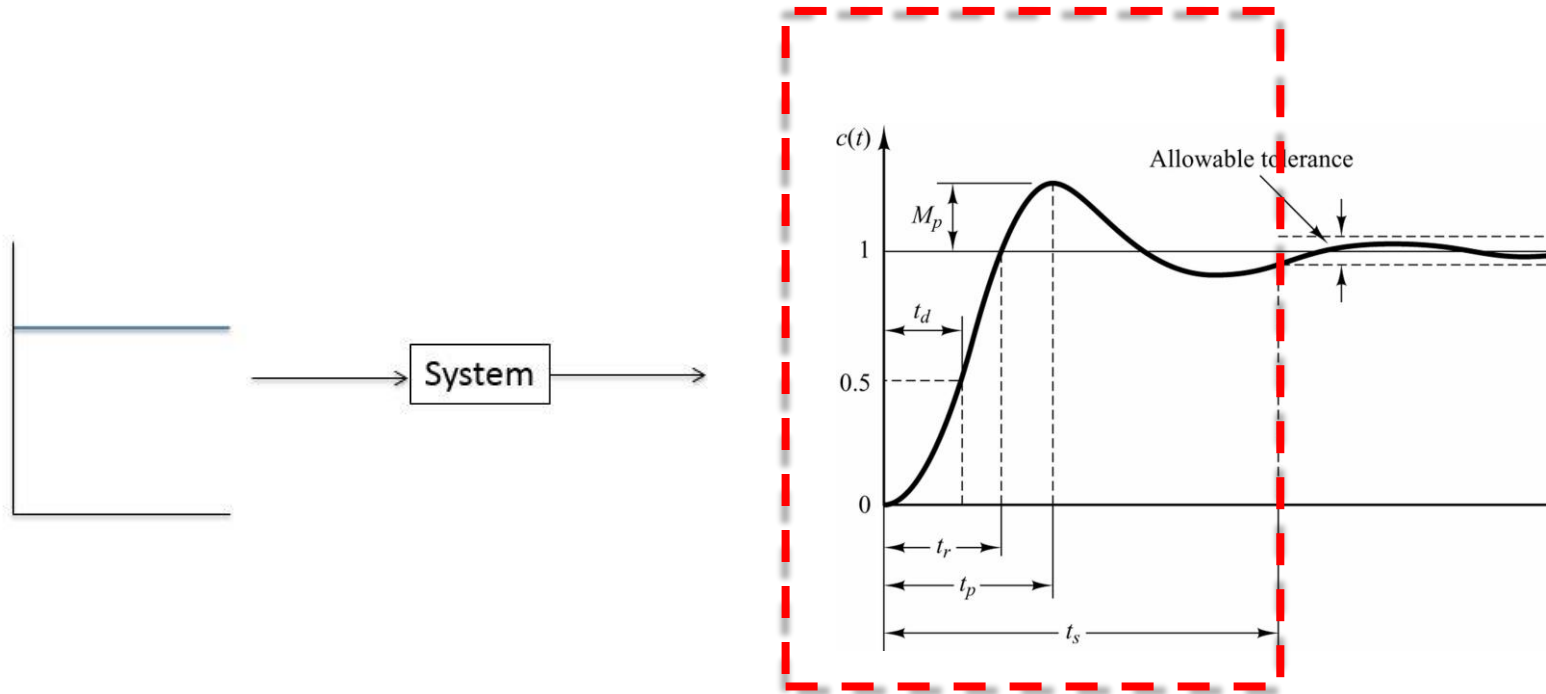
$$y(\infty) = \frac{(A + \Delta A)K_c r_0}{1 + (A + \Delta A)K_c} = \frac{AK_c}{1 + AK_c} \frac{1 + \frac{\Delta A}{A}}{1 + \frac{AK_c}{1 + AK_c} \frac{\Delta A}{A}} r_0$$

- Choosing $AK_c \gg 1$ and keeping only the 1st order term in $\frac{\Delta A}{A}$

$$y(\infty) \cong \left(1 + \frac{1}{1 + AK_c} \frac{\Delta A}{A}\right) r_0$$

- $e_c(\infty) = \frac{\Delta A}{A} \frac{r_0}{1 + AK_c} \ll \frac{\Delta A}{A} r_0 = e_o(\infty) \quad (AK_c \gg 1)$

Step Input Transient Response



Can we control it to get a
“good” transient response?

Open-Loop Control

$$Y(s) = \frac{AK_o r_0}{s(\tau_1 s + 1)(\tau_2 s + 1)} = r_0 \left[\frac{1}{s} - \frac{B_1}{s + \frac{1}{\tau_1}} - \frac{B_2}{s + \frac{1}{\tau_2}} \right]$$



$$y(t) = \left(1 - B_1 e^{-\frac{t}{\tau_1}} - B_2 e^{-\frac{t}{\tau_2}} \right) r_0$$

Time constants τ_1, τ_2 are fixed by the plant model

→ Transient are fixed by the τ_1, τ_2 values

(cannot be improved)

Closed-Loop Control

$$Y(s) = \frac{AK_c r_0}{s[(\tau_1 s + 1)(\tau_2 s + 1) + AK_c]} = \frac{AK_c r_0}{1 + AK_c} \left(\frac{1}{s} - \frac{C_1}{s - s_1} - \frac{C_2}{s - s_2} \right)$$
$$s_1, s_2 = \frac{-(\tau_1 + \tau_2) \pm \sqrt{(\tau_1 + \tau_2)^2 - 4\tau_1\tau_2(1 + AK_c)}}{2\tau_1\tau_2}$$

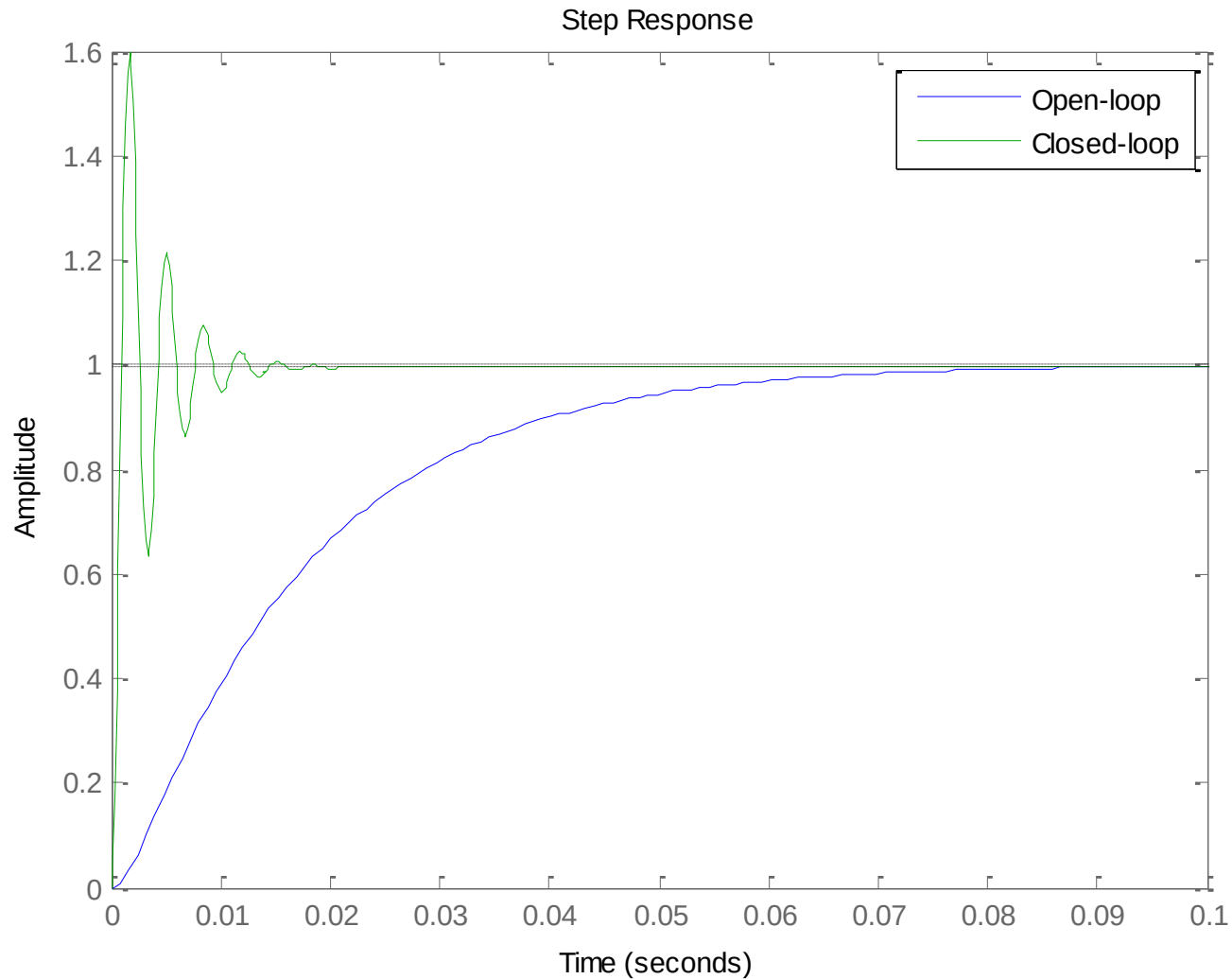


$$y(t) = \frac{AK_c r_0}{1 + AK_c} (1 - C_1 e^{s_1 t} - C_2 e^{s_2 t})$$

- Time constants s_1, s_2 are adjustable by K_c
- $K_c \gg 1 \rightarrow s_1, s_2$ are complex conjugate
 $\rightarrow$ Oscillatory decay if $\tau_1 + \tau_2 > 0$

Transient Responses

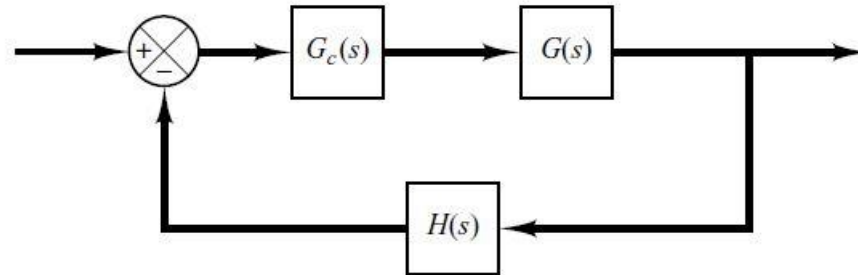
- Unchangeable in open-loop control
- Adjustable to achieve desired output in closed-loop control



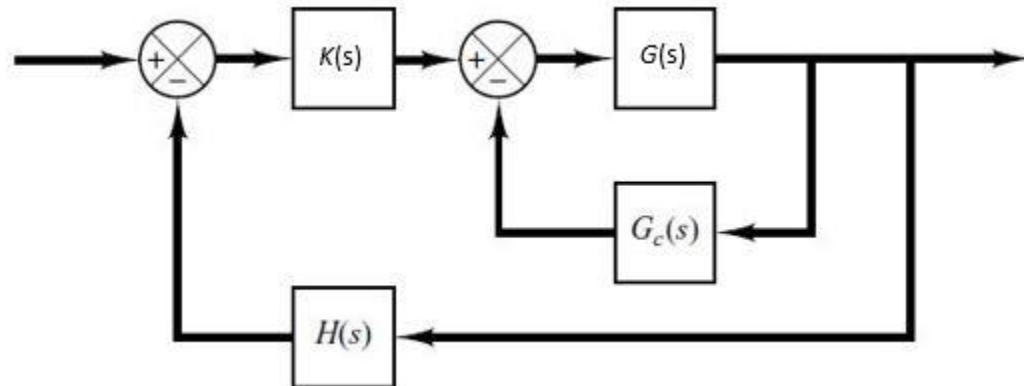
Improving System Performance by Compensation

System transient performance can be improved by using compensator (also called: filter)

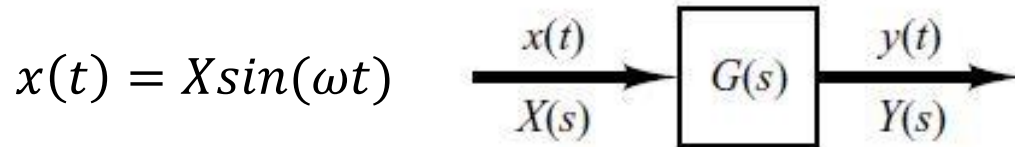
Series:



Parallel:



Consider:



$$G(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{(s + s_1)(s + s_2) \cdots (s + s_n)}$$

$$G(j\omega) = Me^{j\phi} = M \angle \phi$$



$$\begin{aligned} Y(s) &= G(s)X(s) = G(s) \frac{\omega X}{s^2 + \omega^2} \\ &= \frac{a}{s + j\omega} + \frac{\bar{a}}{s - j\omega} + \frac{b_1}{s + s_1} + \frac{b_2}{s + s_2} + \cdots + \frac{b_n}{s + s_n} \end{aligned}$$



$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + b_1e^{-s_1t} + b_2e^{-s_2t} + \cdots + b_ne^{-s_nt}$$

For a stable system:

$$y_{ss}(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + 0 + \cdots + 0$$

$$y_{ss}(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} \quad Y(s) = \frac{a}{s + j\omega} + \frac{\bar{a}}{s - j\omega}$$

$$a = G(s) \frac{\omega X}{s^2 + \omega^2} (s + j\omega) \Big|_{s=-j\omega} = -\frac{XG(-j\omega)}{2j}$$

$$\bar{a} = G(s) \frac{\omega X}{s^2 + \omega^2} (s - j\omega) \Big|_{s=j\omega} = \frac{XG(j\omega)}{2j}$$

$$G(j\omega) = |G(j\omega)|e^{j\phi} \quad \phi = \angle G(j\omega)$$

$$G(-j\omega) = |G(-j\omega)|e^{-j\phi} = |G(j\omega)|e^{-j\phi}$$

$$\Rightarrow a = -\frac{X|G(j\omega)|e^{-j\phi}}{2j}, \quad \bar{a} = \frac{X|G(j\omega)|e^{j\phi}}{2j}$$

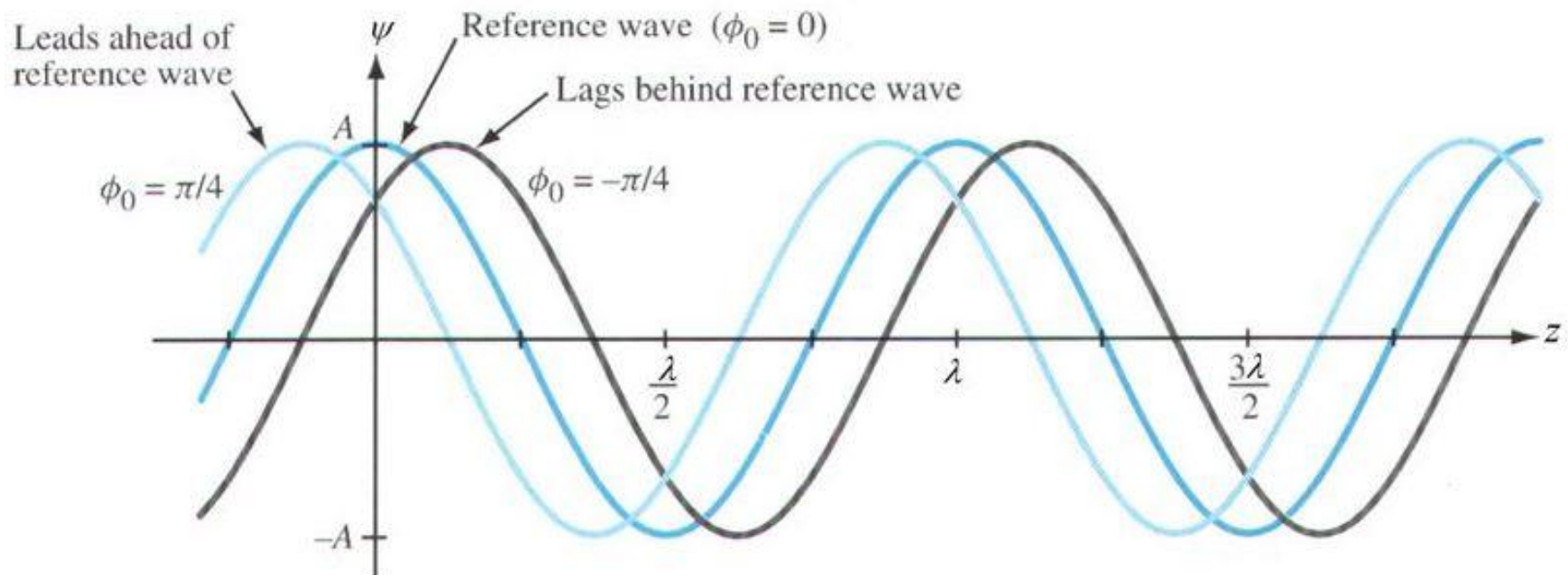
$$\Rightarrow y_{ss}(t) = X|G(j\omega)| \frac{e^{j(\omega t + \phi)} - e^{-j(\omega t + \phi)}}{2j}$$

$$= X|G(j\omega)| \sin(\omega t + \phi)$$

$$= Y \sin(\omega t + \phi)$$

Phase-Lead and Phase-Lag

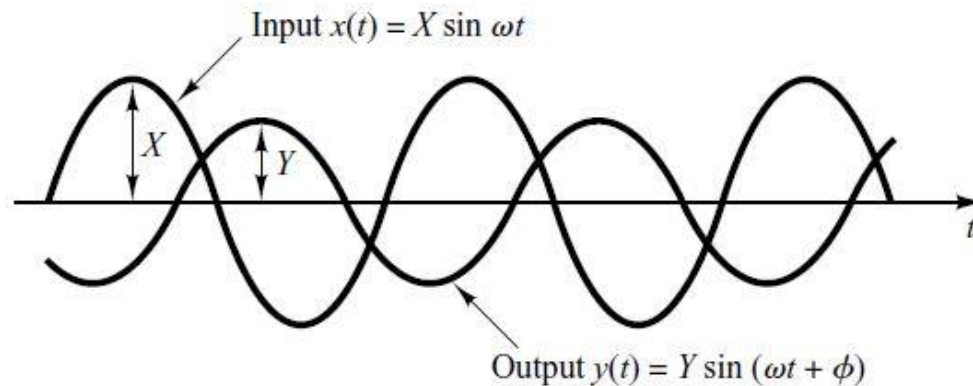
$$\psi(t) = A \cos\left(\frac{2\pi}{\lambda}t + \phi_0\right)$$



Positive angle – Phase lead (ahead)

Negative angle – Phase lag (behind)

sinusoidal input → sinusoidal output



If the steady-state sinusoidal output has a **positive phase** angle ϕ , called **phase lead**, then the system is called a **lead system**

If the steady-state sinusoidal output has a **negative phase** angle ϕ , called **phase lag**, then the system is called a **lag system**

If **both** phase lag and phase lead occur in the output but in different frequency regions (usually, phase lag occurs in the low-frequency region and phase lead occurs in the high-frequency region), the system is called a **lag-lead system**

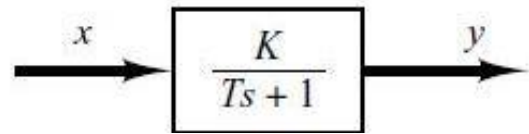
In general, the amount of phase shift angle ϕ is a function of the input frequency ω

A compensator having a characteristic of a lead system, lag system, or lag–lead system is called a **lead compensator**, **lag compensator**, or **lag–lead compensator**, respectively.

Widely used compensators are: lead compensators, lag compensators, lag–lead compensators, and velocity-feedback (tachometer) compensators.

Lead, lag, and lag–lead compensators may be electronic devices (such as circuits using operational amplifiers) or RC networks (electrical, mechanical, pneumatic, hydraulic, or combinations thereof) and amplifiers.

Example:



$$G(s) = \frac{K}{Ts+1} \rightarrow G(j\omega) = \frac{K}{jT\omega+1}$$

$$|G(j\omega)| = \frac{K}{\sqrt{1 + T^2\omega^2}}$$

$$\phi = \angle G(j\omega) = -\tan^{-1} T\omega$$

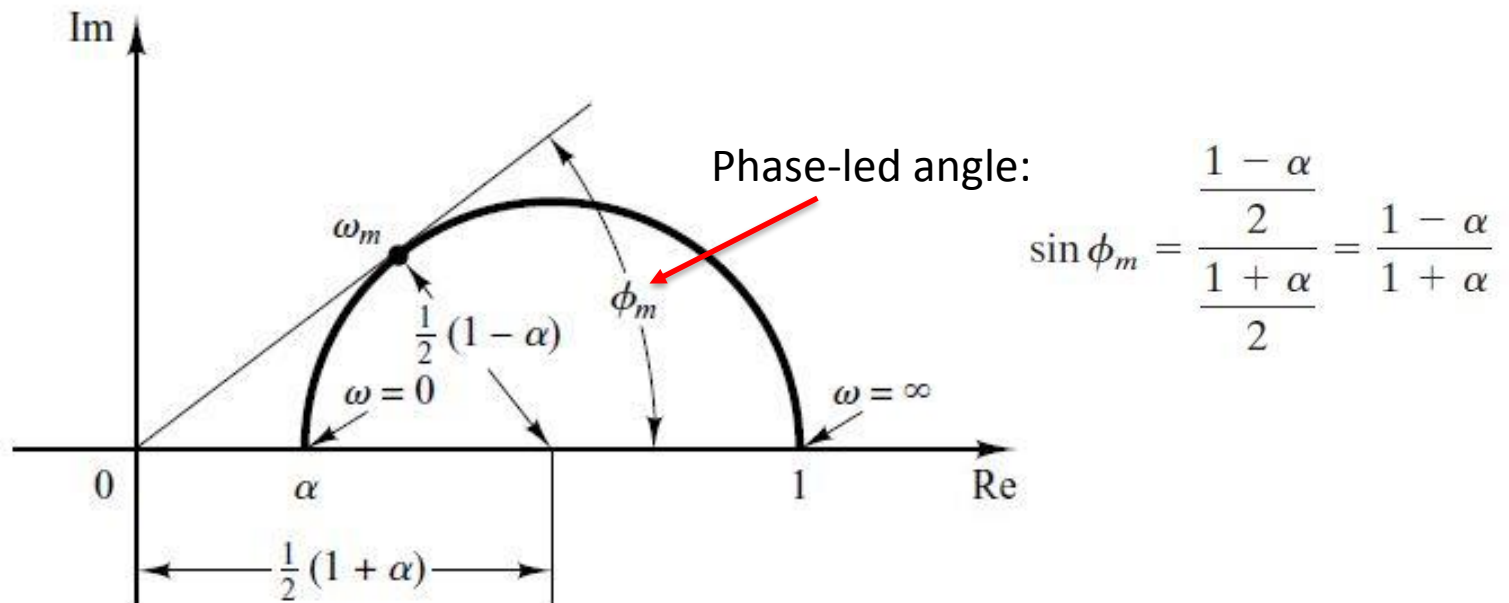
→
$$y_{ss}(t) = \frac{XK}{\sqrt{1 + T^2\omega^2}} \sin(\omega t - \tan^{-1} T\omega)$$

Phase shift $\phi \rightarrow -90^\circ$ as $\omega \rightarrow \infty$

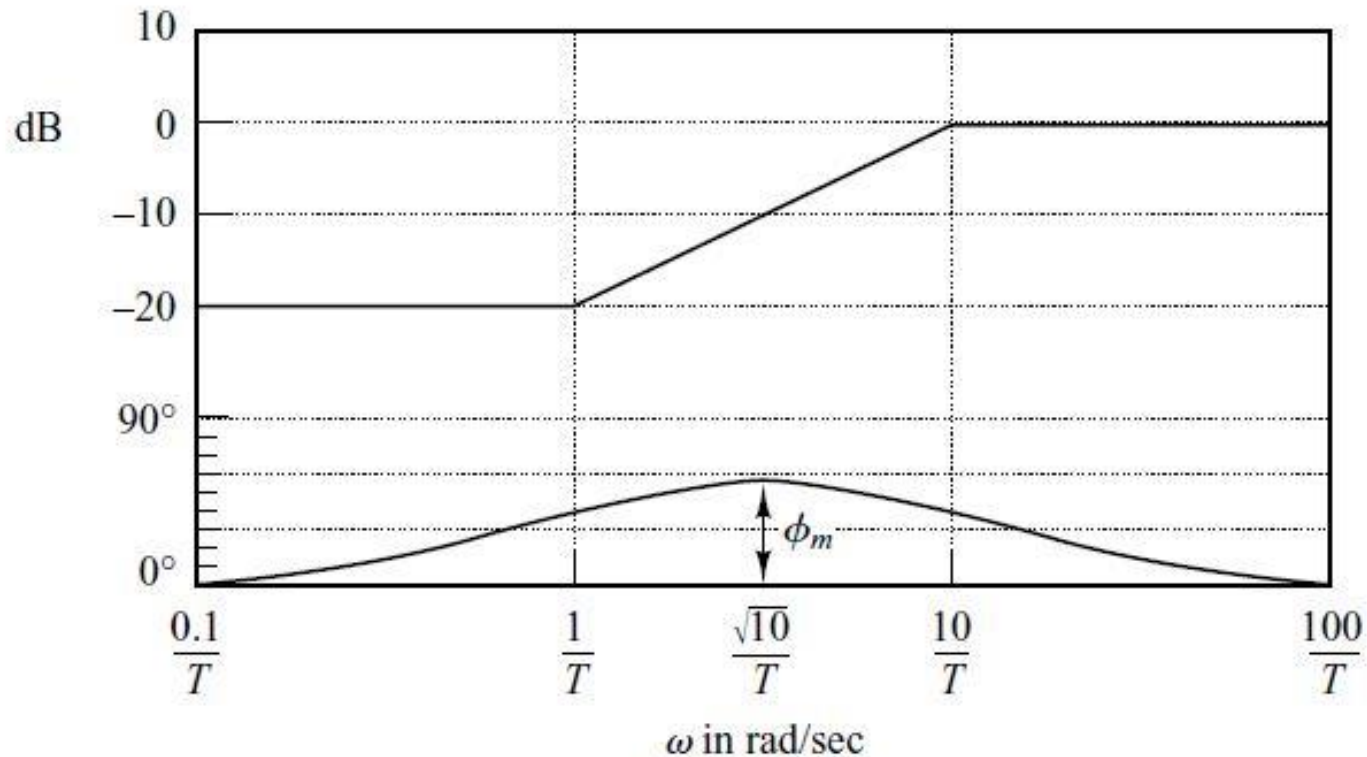
This is a **phase-lag** compensator

Example (polar plot):

$$G(j\omega) = \alpha \frac{j\omega T + 1}{j\omega\alpha T + 1} \quad (0 < \alpha < 1)$$



Bode plot ($\alpha = 0.1$)

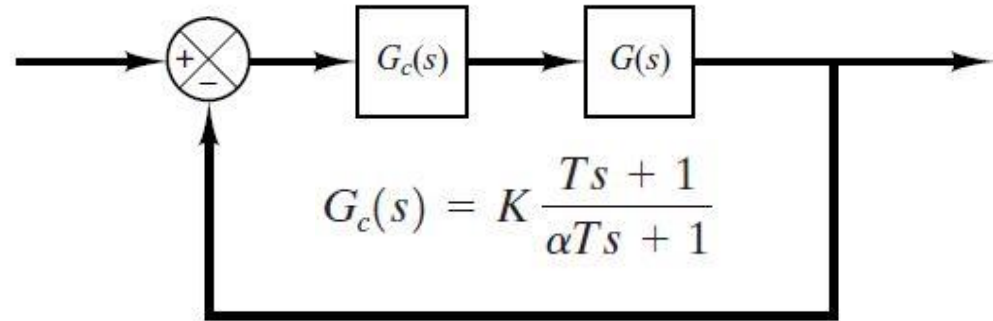


ϕ_m is geometric
mean of the two
corner frequencies
 $1/T$ and $10/T$

$$\Rightarrow \log \omega_m = \frac{1}{2} \left(\log \frac{1}{T} + \log \frac{1}{\alpha T} \right) \Rightarrow \omega_m = \frac{1}{\sqrt{\alpha} T}$$

Lead Compensation

Example:

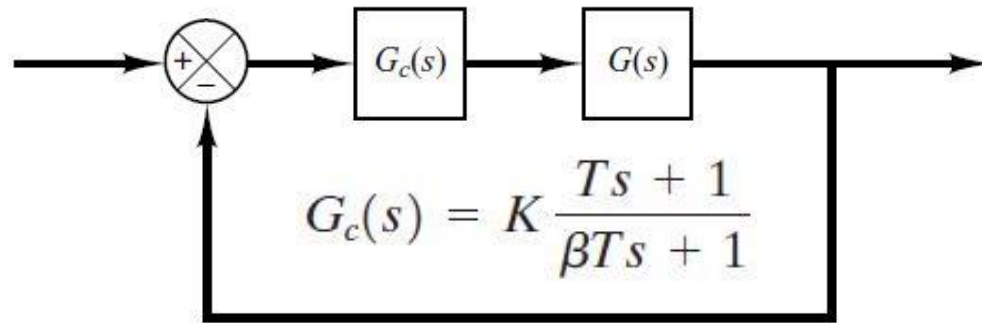


$$G_c(s)G(s) = K \frac{Ts + 1}{\alpha Ts + 1} G(s) = \frac{Ts + 1}{\alpha Ts + 1} KG(s) \quad (K = K_c \alpha)$$

- Determine K to satisfy the required steady-state position y_{ss}
- Determine the phase-lead angle ϕ_m to be added to the system
- Draw the Bode plot of $KG(j\omega)$, and compute α by $\sin \phi_m = \frac{1-\alpha}{1+\alpha}$
- Determine ω where magnitude of un-compensated system $|KG(j\omega)| = -20 \log(\frac{1}{\sqrt{\alpha}})$
- Input phase shift which yields maximum phase shift, is $\omega_m = 1/(\sqrt{\alpha} T)$
- Repeat the above until the output phase-lead $\approx$ required ϕ_m
- Compensator gain $K_c = \frac{K}{\alpha}$

Lag Compensation

Example:

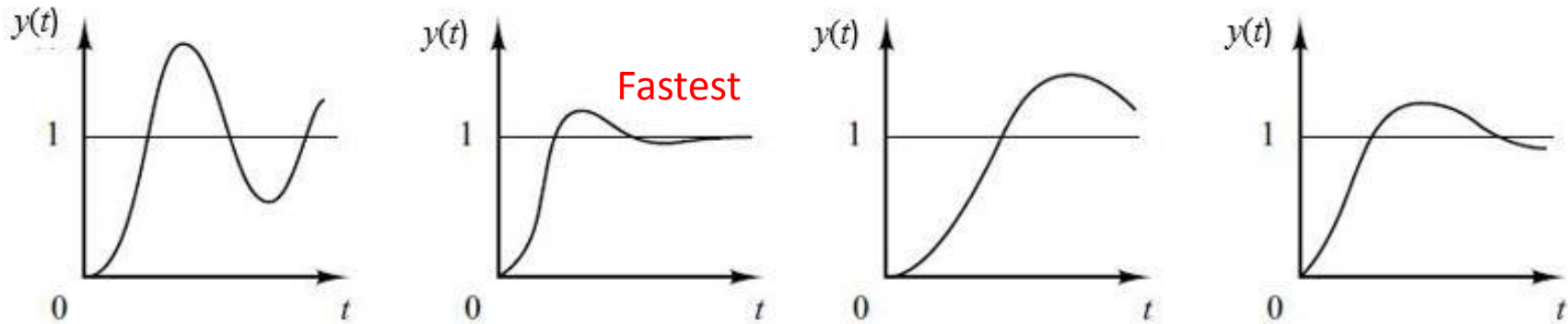


$$G_c(s)G(s) = K \frac{Ts + 1}{\beta Ts + 1} G(s) = \frac{Ts + 1}{\beta Ts + 1} KG(s) \quad (K = K_c\beta)$$

- Determine K to satisfy the required steady-state velocity error $\dot{y}_{ss}$
- Determine the phase-lag angle ϕ_m to be added to the system
- Draw the Bode plot of $KG(j\omega)$, and determine $\omega_m = -180^\circ + \phi_m$
- Determine β to bring the magnitude curve down to 0 dB
- Input phase shift, which yields maximum phase shift, is $\omega_m = 1/(\beta T)$
- Repeat the above until the output phase-lag $\approx$ required ϕ_m
- Compensator gain $K_c = \frac{K}{\beta}$

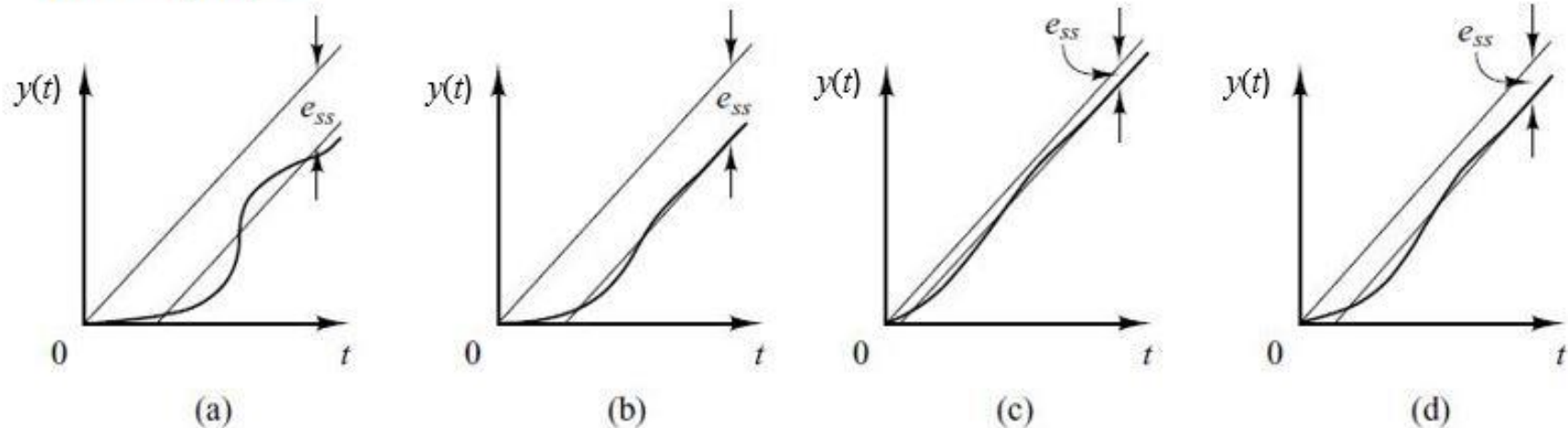
Graphical Comparison

Unit step input



Fastest

Unit ramp input



Smallest error

Compromise

(a) un-compensated (b) lead-compensated (c) lag-compensated (d) Lead-lag compensated

END