

Complex Networks:

Lecture 8: Network Synchronization

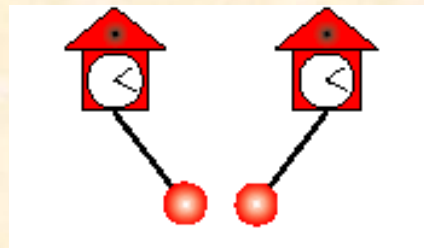
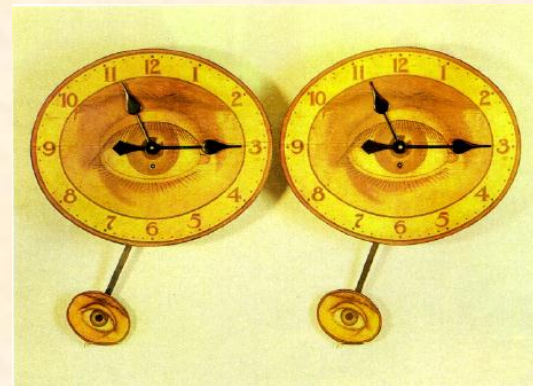
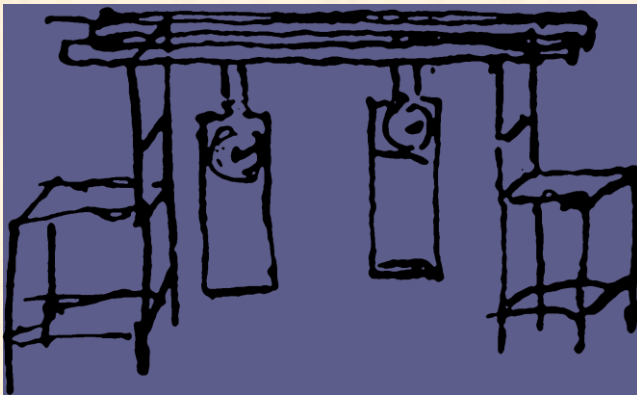
EE 6605

Instructor: G Ron Chen



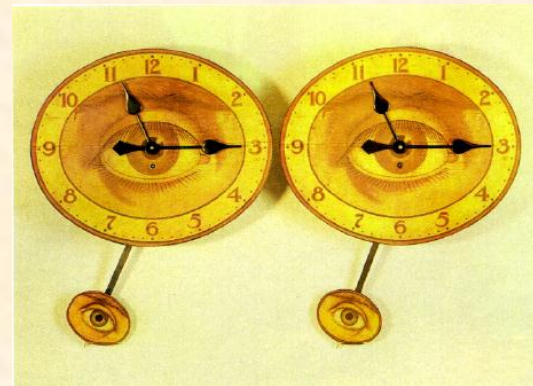
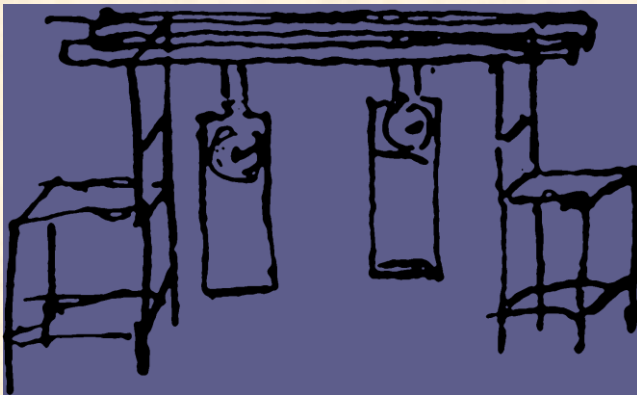
Synchronization

(Huygens, 1665)



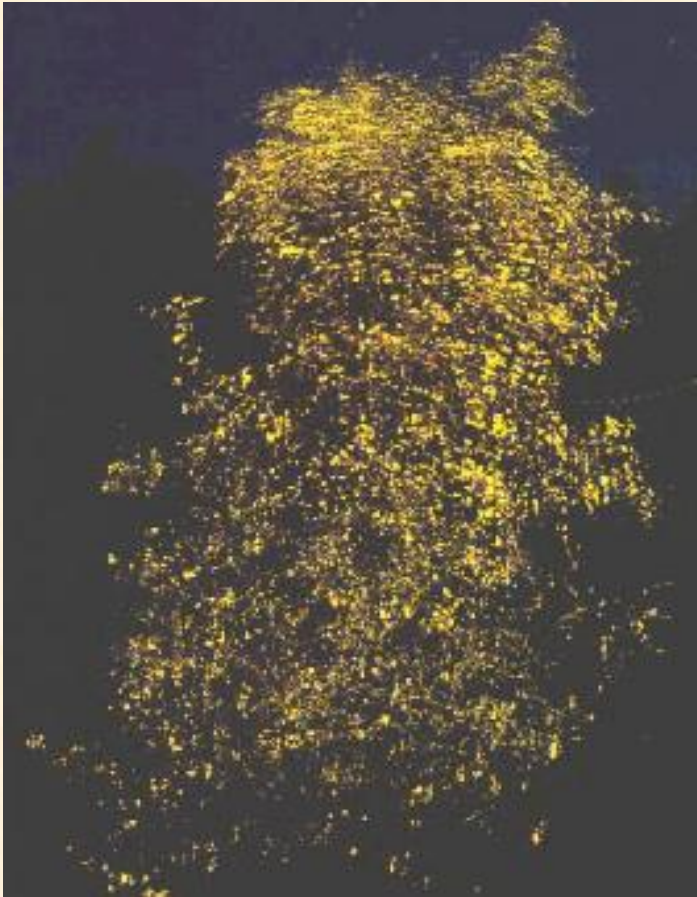
Synchronization

(Huygens, 1665)

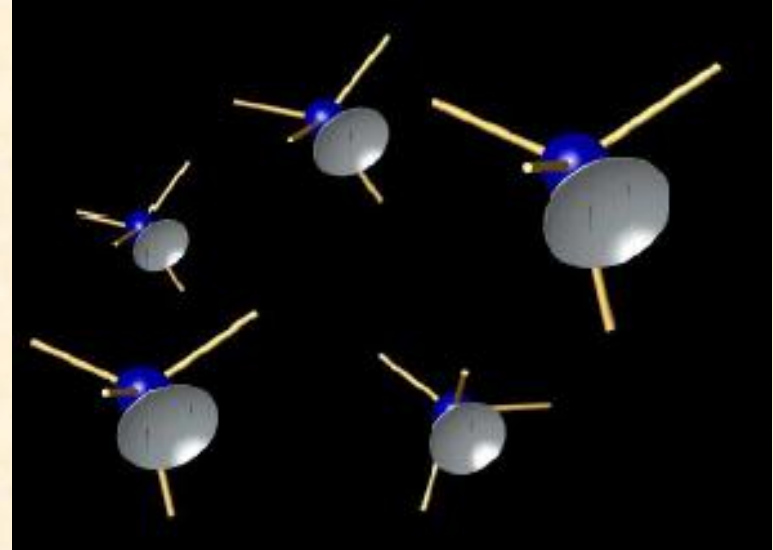


DEMO

Fireflies Synchronization



Attitude Alignment



The attitude of each spacecraft is synchronized with its two adjacent neighbors via a bi-directional communication channel

Synchronization-based Laser

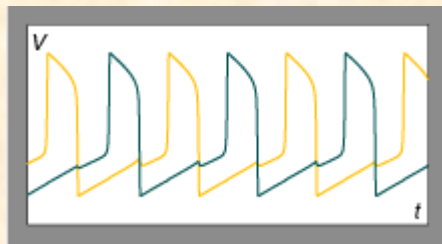
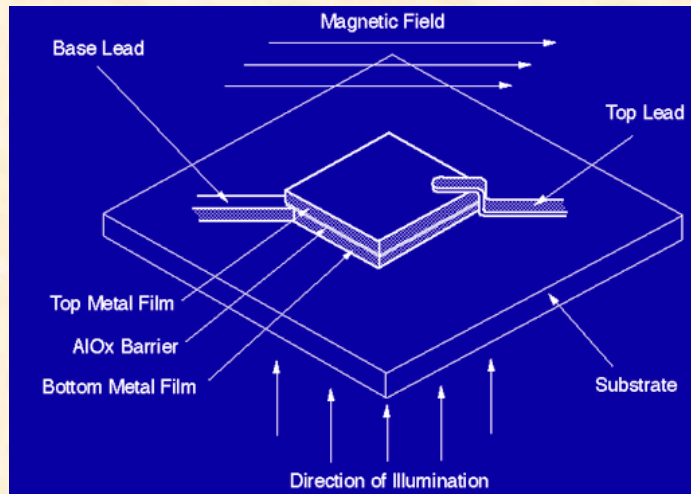


Astounding coherence of a laser beam comes from trillions of atoms pulsing in concert, all emitting photons of the same phase and frequency

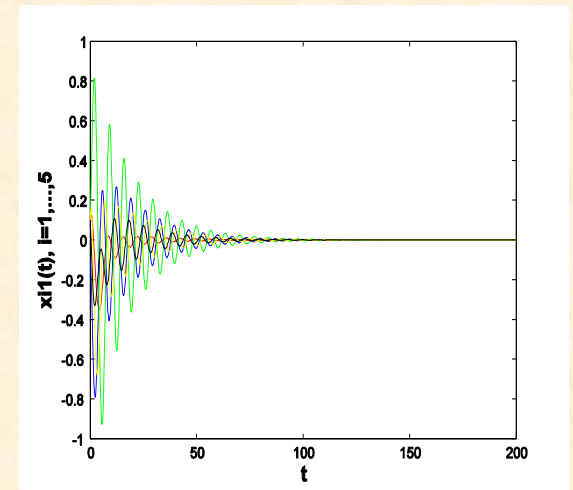
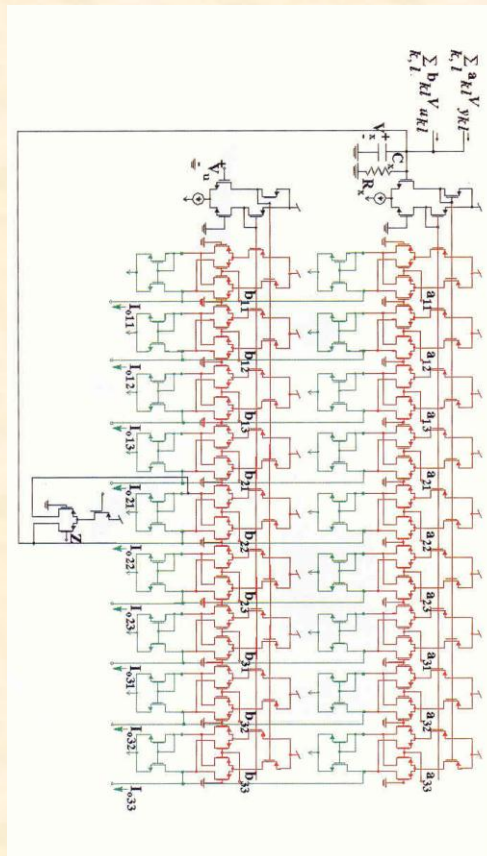
Electric Currents

Through Josephson Junctions

Oscillate as One



Phase Sync
DEMO



Simulation

Synchronization in Neural Systems



B. Ibarz ^a, J.M. Casado ^b, M.A.F. Sanjuán, Map-based models in neuronal dynamics, *Physics Reports*, 501 (2011) 1-74.

"Synchronous behavior of neural assemblies is a central topic in neuroscience. It is known to be correlated with cognitive activities [109] during the normal functioning of the brain, while **abnormal synchronization** is linked to important **brain disorders**, such as epilepsy, Parkinson's disease, Alzheimer's disease, schizophrenia and autism [110]. Hence the interest in the topic of **neural synchronization** has been extensively explored theoretically [111]."

[109] W. Singer, Neuronal synchrony: a versatile code for the definition of relations? *Neuron* 24 (1999) 49–65.

[110] P.J. Uhlhaas, W. Singer, Neural synchrony in brain disorders: relevance for cognitive dysfunctions and pathophysiology, *Neuron* 52 (2006) 155–168.

[111] H.D.I. Abarbanel, M.I. Rabinovich, A. Selverston, M.V. Bazhenov, R. Huerta, M.M. Suschchik, L.L. Rubchinskii, Synchronisation in neural networks, *Phys. Usp.* 39 (1996) 337–362.

Out of tumultuous applause: Synchronized claps

Self-organization
in the concert-hall:
the dynamics of
rhythmic applause

--- *Nature* (2001)



Pedestrians Make London's Millennium Bridge Wobble

June 10, 2000



[Video](#)

Birds Flocking



Flocking: to congregate or travel in flock

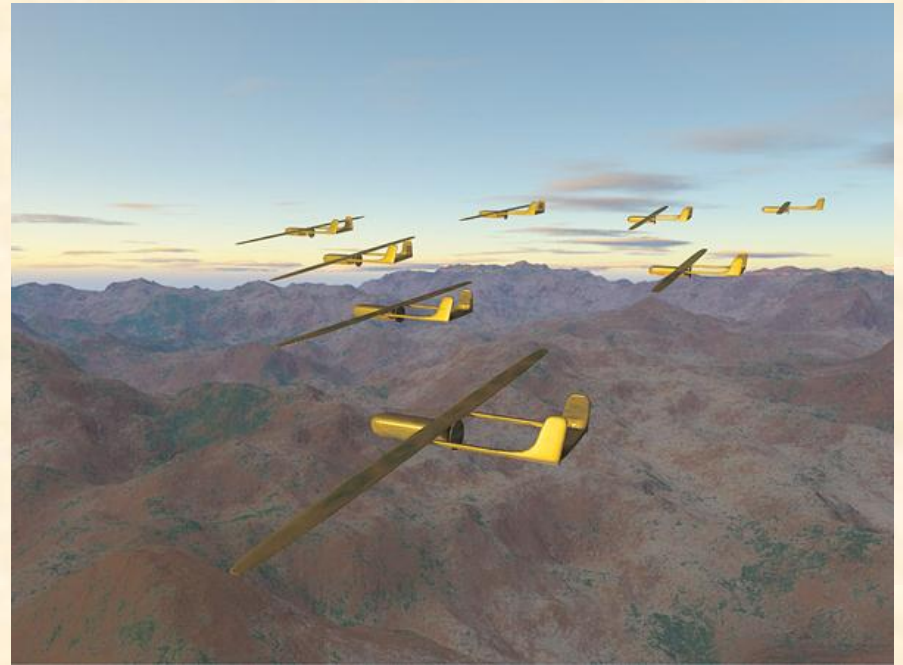
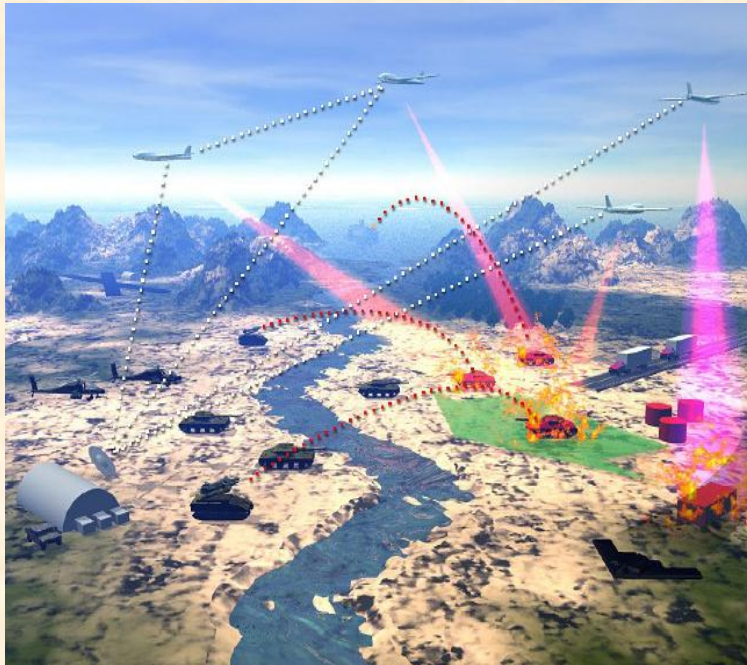
Fish Swarming

- ❖ **Swarming:**
to move or gather in group



Consensus

A position reached by a group as a whole



Battle space management scenario

Harmful Synchronization in Internet

- **TCP window increase/decrease cycles**

Synchronization occurs when separate TCP connections share a common bottleneck router

- **Synchronization to an external clock**

Two processes can become synchronized simply if they are both synchronized to the same external clock

- **Client-server models**

Multiple clients can become synchronized as they wait for services from a busy (or recovering) server

- **Periodic message routing**

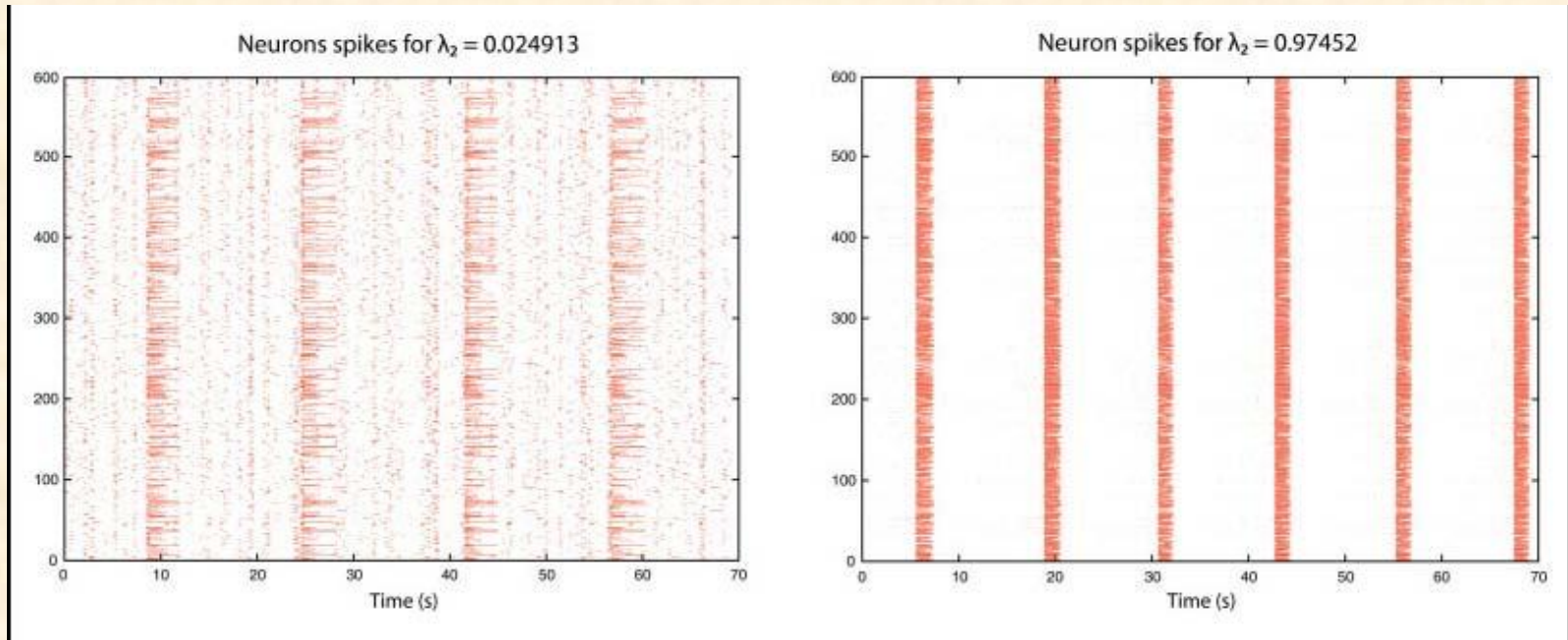
- **.....**

Useful Synchronization in Engineering

Examples:

- Secure communications
- Generation of harmonic oscillations
- Language emergence and development:
 - Synchronization in conversations
 - common vocabulary
- Organization management:
 - Agents' synchronization
 - work efficiency
- Biological systems (brain, heart)
-

One Example



non-synchronous

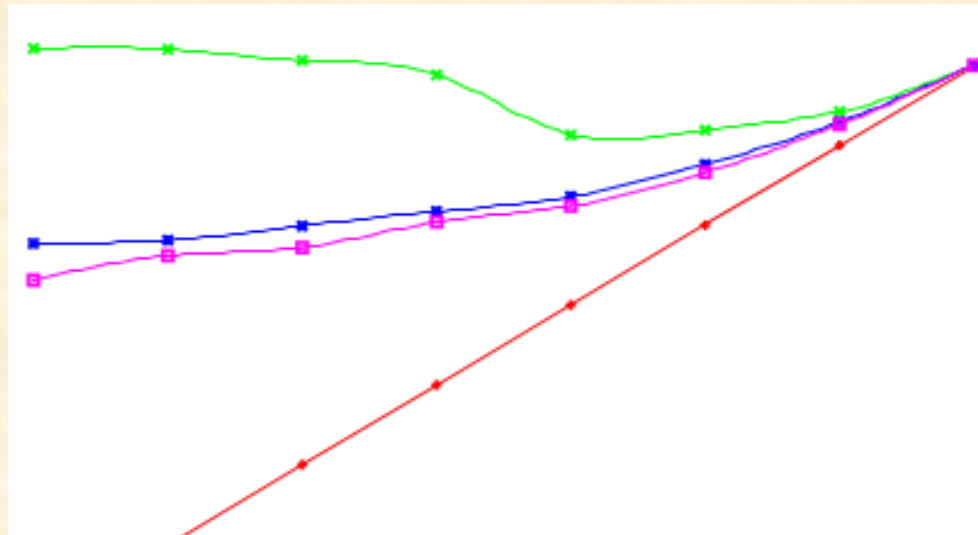
synchronous

In mammals, a small group of neurons in the brain stem, named pre-Bötzinger complex, is responsible for generating a regular rhythmic output to motor calls that initiate a breath.

Exploring Synchronization

First, how to describe the individual dynamics of each isolated node?

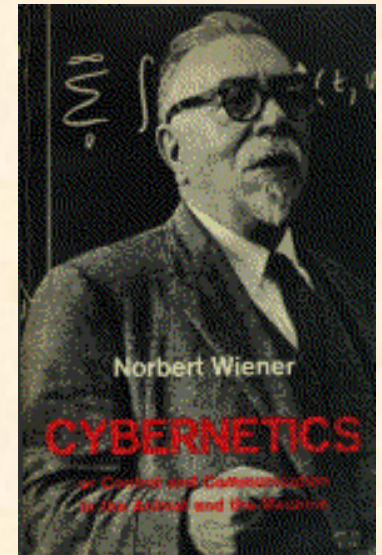
Then, how to describe the synchronous dynamics of the coupled network?



Norbert Wiener

(1894-1964)

- ❖ **Collective synchronization** was first studied mathematically by **Wiener**, who recognized its ubiquity in the natural world, and speculated that it was involved in the generation of alpha rhythms in the brain.
- ❖ **Unfortunately**, Wiener's mathematical approach based on Fourier integrals has turned out to be a dead end.
- ❖ N. Wiener, **Nonlinear Problems in Random Theory**, MIT Press, 1958
- ❖ N. Wiener, **Cybernetics**, MIT Press, 1st ed., 1948, 2nd ed., 1961.



Network Synchronization

- Network Synchronization versus Individual Dynamics
 - Typical networks: coupled map lattices, cellular neural networks, complex networks
 - Typical dynamics: Turing patterns, autowaves, spiral waves, spatiotemporal chaos
- Network topology and the dynamics of individual nodes determine the network dynamical behaviors
 - Synchronization

A General Dynamical Network Model

Linearly and diffusively coupled:

$$\dot{x}_i = f(x_i) + c \sum_{j=1}^N a_{ij} H(x_j) \quad x_i = \begin{bmatrix} x_i^1 \\ x_i^2 \\ \vdots \\ x_i^n \end{bmatrix} \in R^n \quad i = 1, 2, \dots, N$$

dynamics: $f(\cdot)$ – Lipschitz function

coupling strength $c > 0$

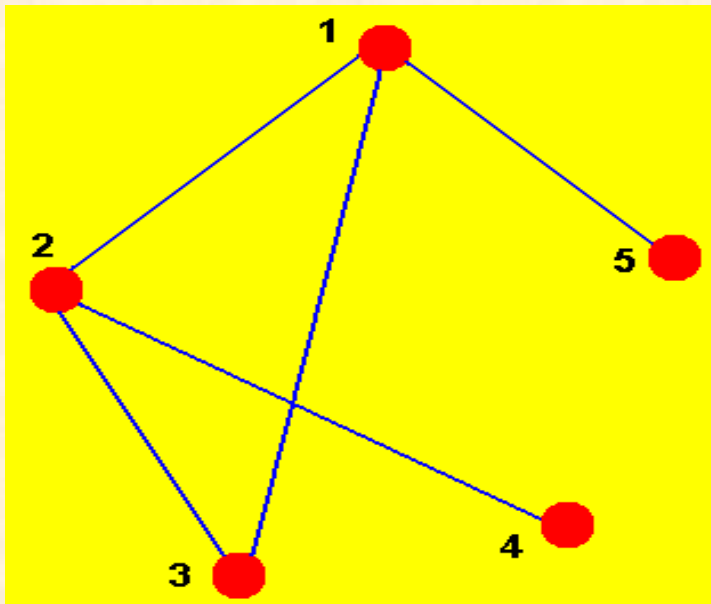
coupling matrices $A = [a_{ij}]_{N \times N}$ $H(x_j) = \begin{bmatrix} H_1(x_j) \\ H_2(x_j) \\ \vdots \\ H_n(x_j) \end{bmatrix}$ e.g. $H = \Gamma = \begin{bmatrix} r_{11} & & & 0 \\ & r_{22} & & \\ & & \ddots & \\ 0 & & & r_{nn} \end{bmatrix}$

If there is a connection between node i and node j ($j \neq i$),
then $a_{ij} = a_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0$; And $a_{ii} = 0$

$$D = [d_{ii}] = \text{diag}\{k_1, \dots, k_N\}$$

Laplacian matrix $L = D - A$ (zero row sum: diffusive coupling)

Simple Examples

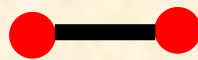


$$L = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

symmetrical, diffusive,
irreducible, zero row-sum

eigenvalues of L :

$$\lambda_1 = 0 < \lambda_{2,3} = \frac{5}{2} \pm \frac{\sqrt{5}}{2}, \lambda_{4,5} = \frac{5}{2} \pm \frac{\sqrt{13}}{2}$$



$$L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

eigenvalues of L : $\lambda_1 = 0 \quad \lambda_2 = 2$

Network Synchronization: Definition

$$\dot{x}_i = f(x_i) + c \sum_{j=1}^N a_{ij} H(x_j) \quad x_i \in R^n \quad i = 1, 2, \dots, N$$

Complete state
synchronization:

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t)\|_2 = 0, \quad i, j = 1, 2, \dots, N$$

State tracking:

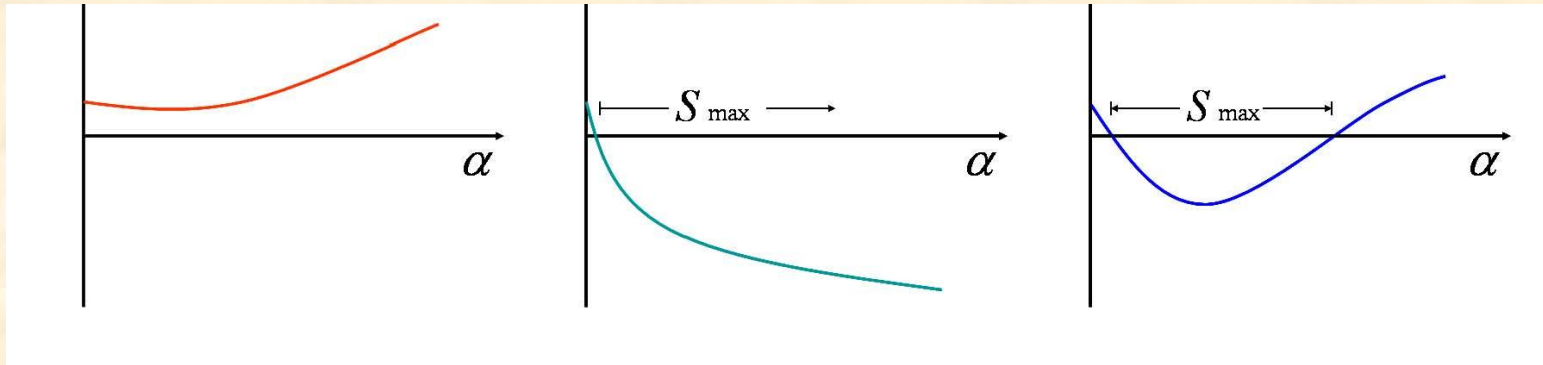
$$x_i(t) \rightarrow s(t) \quad i = 1, 2, \dots, N$$

$$\dot{s}(t) = f(s(t)) \quad (\text{e.g., equilibrium point, periodic orbit, chaotic orbit})$$

$$s(t) \in R^n \quad \text{-- Synchronized state}$$

Synchronization: Criteria

Recall: Eigenvalues of L $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$



synchronizing if $0 < c\lambda_2 < \alpha < \infty$ or if $0 < \frac{\lambda_2}{\lambda_N} < \alpha$

Case I:
No sync

Case II:
Sync region

$$S_1 = (\alpha_1, \infty)$$

Case III:
Sync region

$$S_2 = (\alpha_2, \alpha_3)$$

Case IV:
Union of intervals

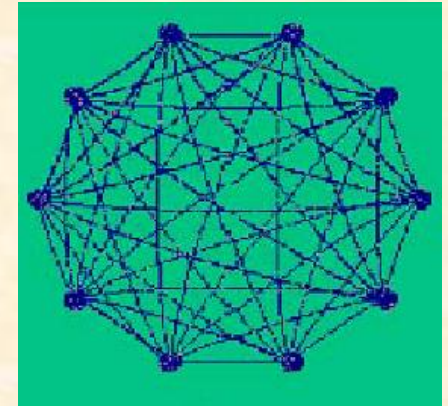
λ_2 bigger is better

$\frac{\lambda_2}{\lambda_N}$ bigger is better

Synchronization in Regularly Coupled Networks

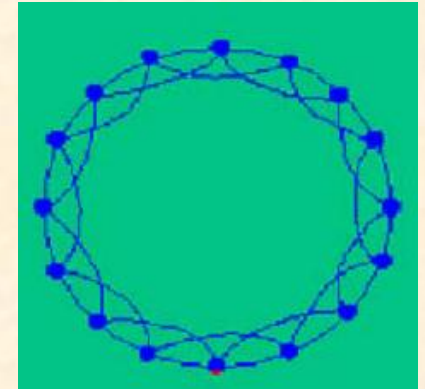
Globally coupled network: $\lambda_2 = \dots = \lambda_N = N$

Usually, no matter how small the coupling strength c is, a global coupled network will synchronize if its size is sufficiently large



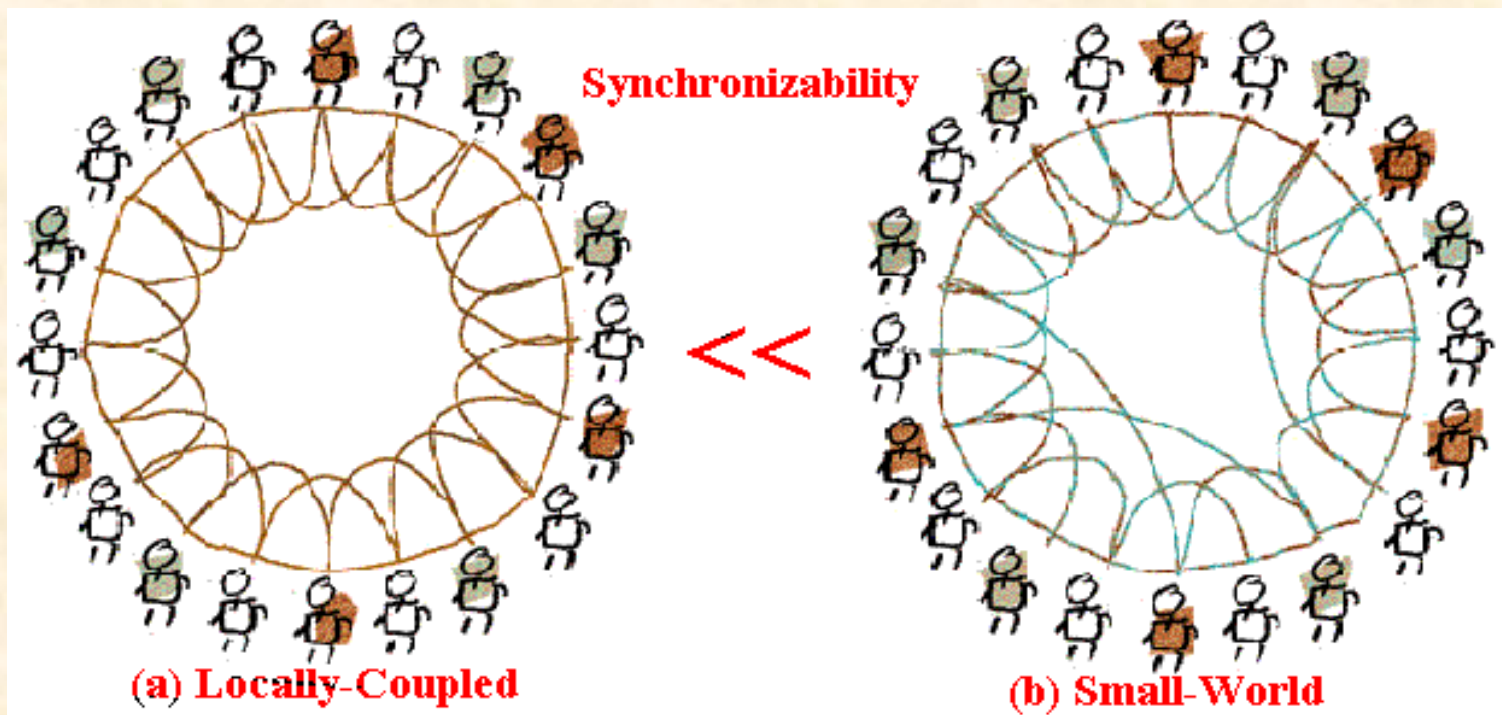
Locally coupled network: $\lambda_2 = 4 \sum_{j=1}^{K/2} \sin^2(j\pi / N)$

Usually, no matter how large the coupling strength c is, a locally coupled network will not synchronize if its size is too large

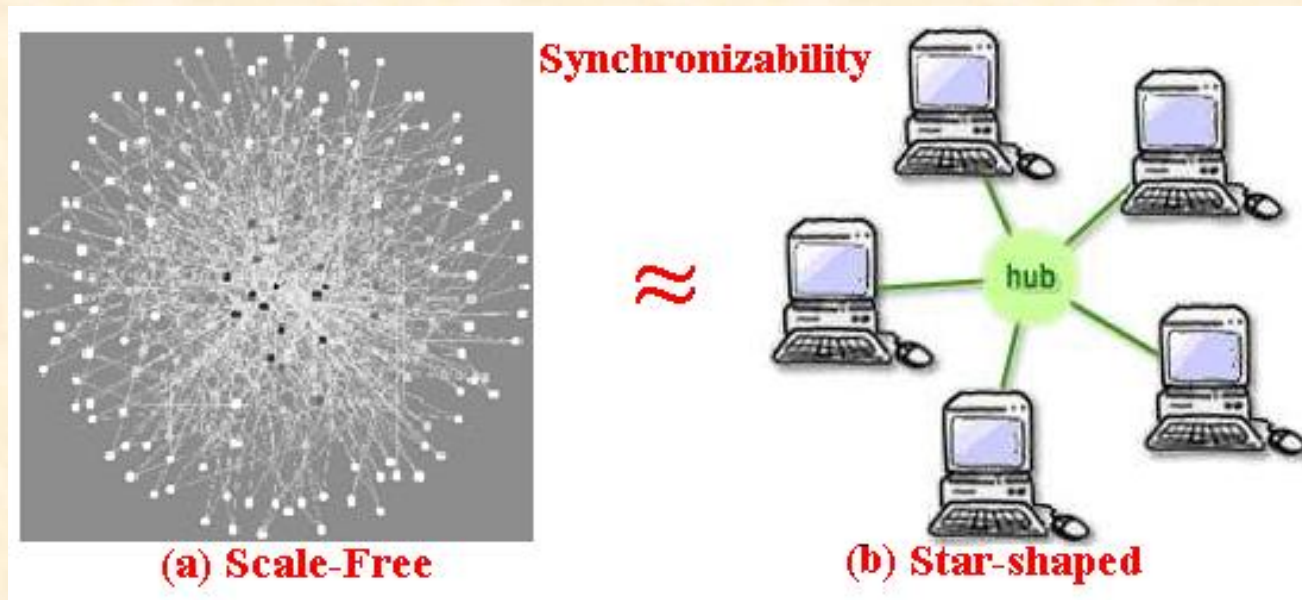


Synchronization in Small-World Networks

Synchronizability can be greatly enhanced by adding a tiny fraction of long-range connections, revealing an advantage of small-world network for synchronization



Synchronization in Scale-Free Networks

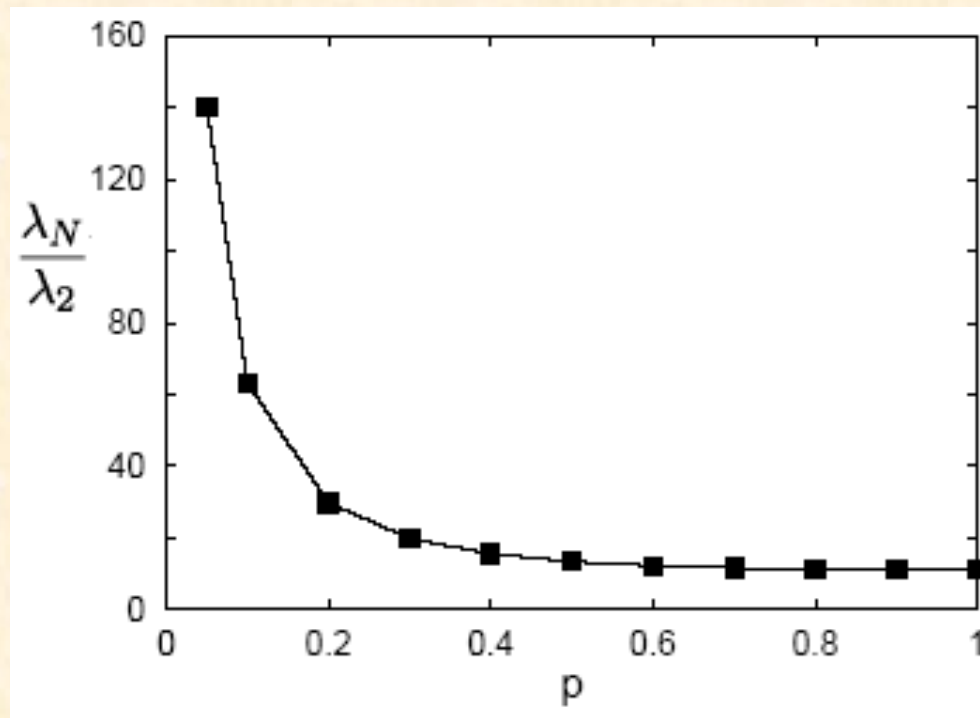


- The **synchronizability** of a SF network is about the same as that of a star-shaped network, mainly determined by the (one-to-one) relations between the central node and each of the neighboring nodes
- This is due to the extremely inhomogeneous connectivity distribution of an SF network: a few hubs in an SF network play a similar role as a single center in a star-shaped network

Network Topology versus Synchronizability

Connection Probability:

Synchronizability of small-world networks will increase as the small-world feature increases



Larger p



Smaller λ_N / λ_2
(Criterion II)



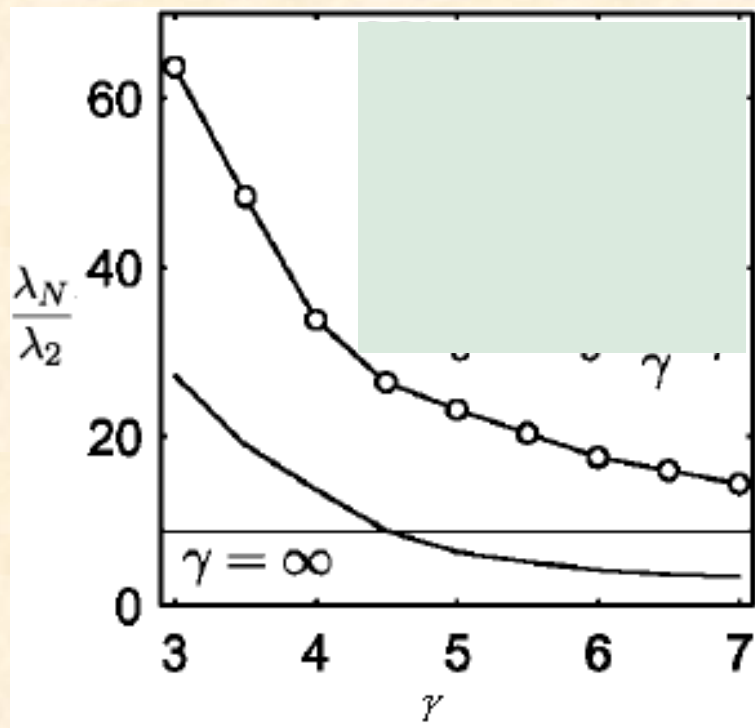
Better synchrony

$p = 1 \rightarrow$ fully connected

Network Topology versus Synchronizability

Power-Law Exponent:

Synchronizability of scale-free networks will increase as the power-law exponent increases



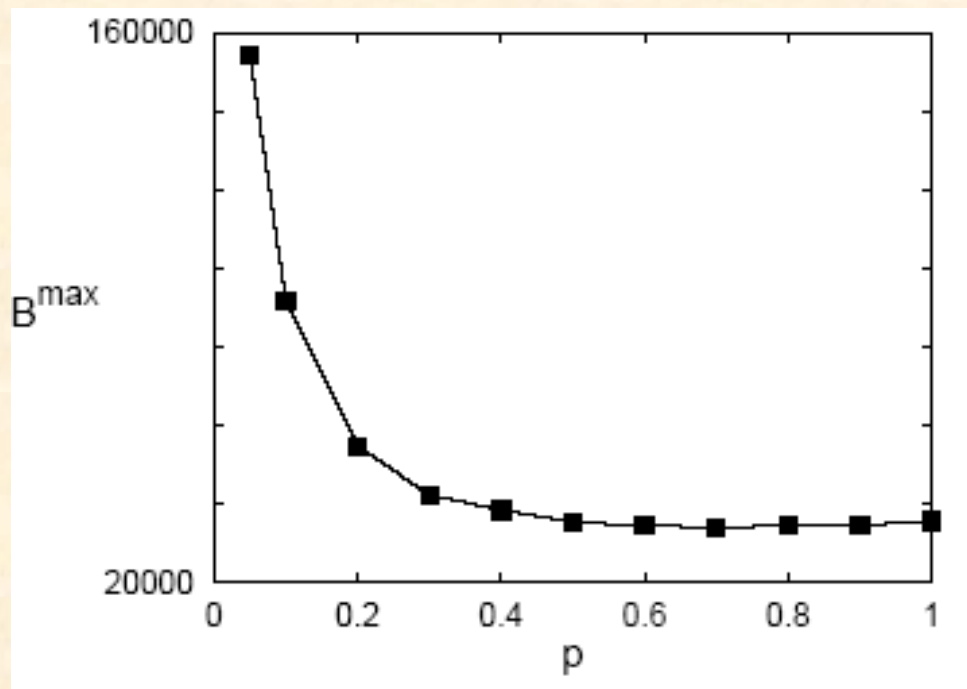
Larger γ
→
Smaller λ_N / λ_2
(Criterion II)
→
Better synchrony

Nishikawa, et al. (2003)

Network Topology versus Synchronizability

Betweenness Centrality:

Synchronizability of small-world and Scale-Free networks will increase as the node betweenness decreases



Larger p



Smaller
node betweenness

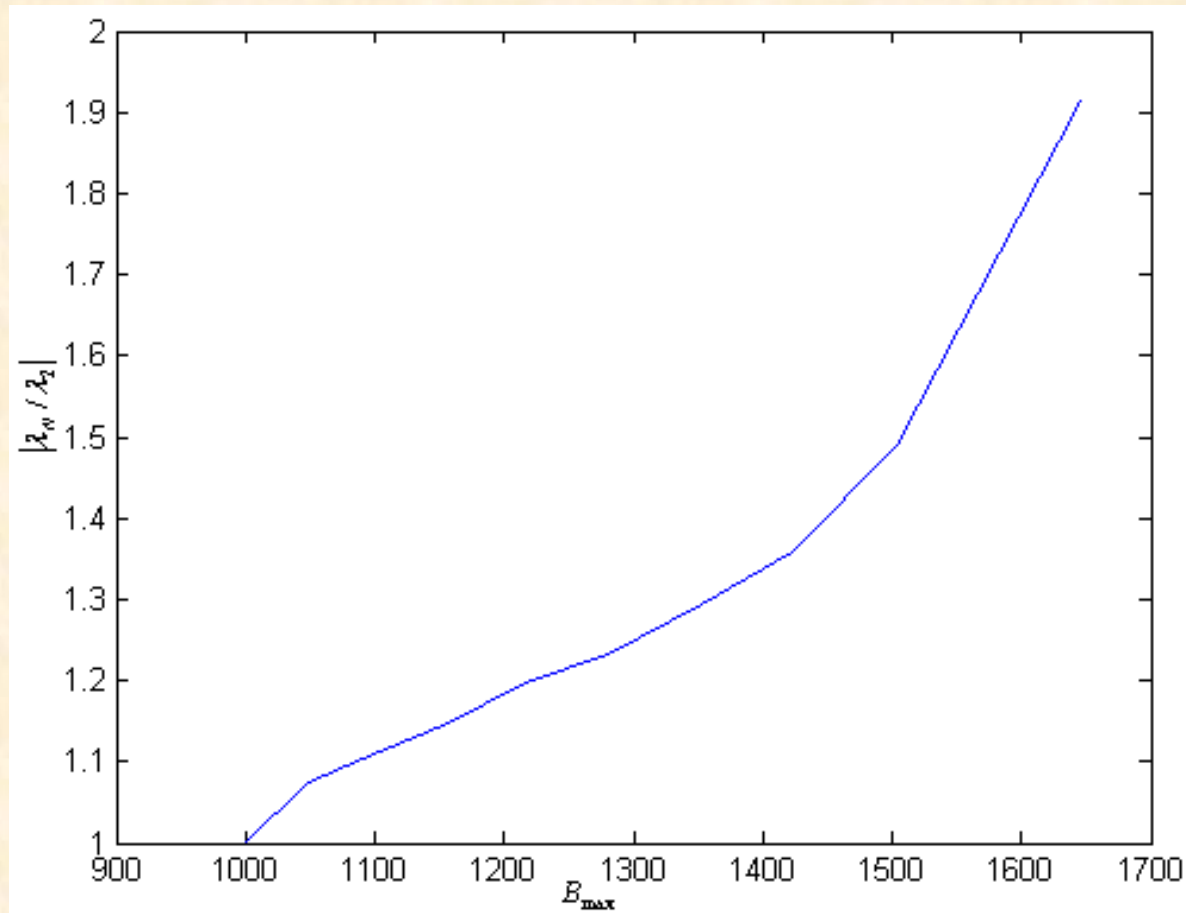


Better synchrony

($p = 1 \rightarrow$ fully connected)

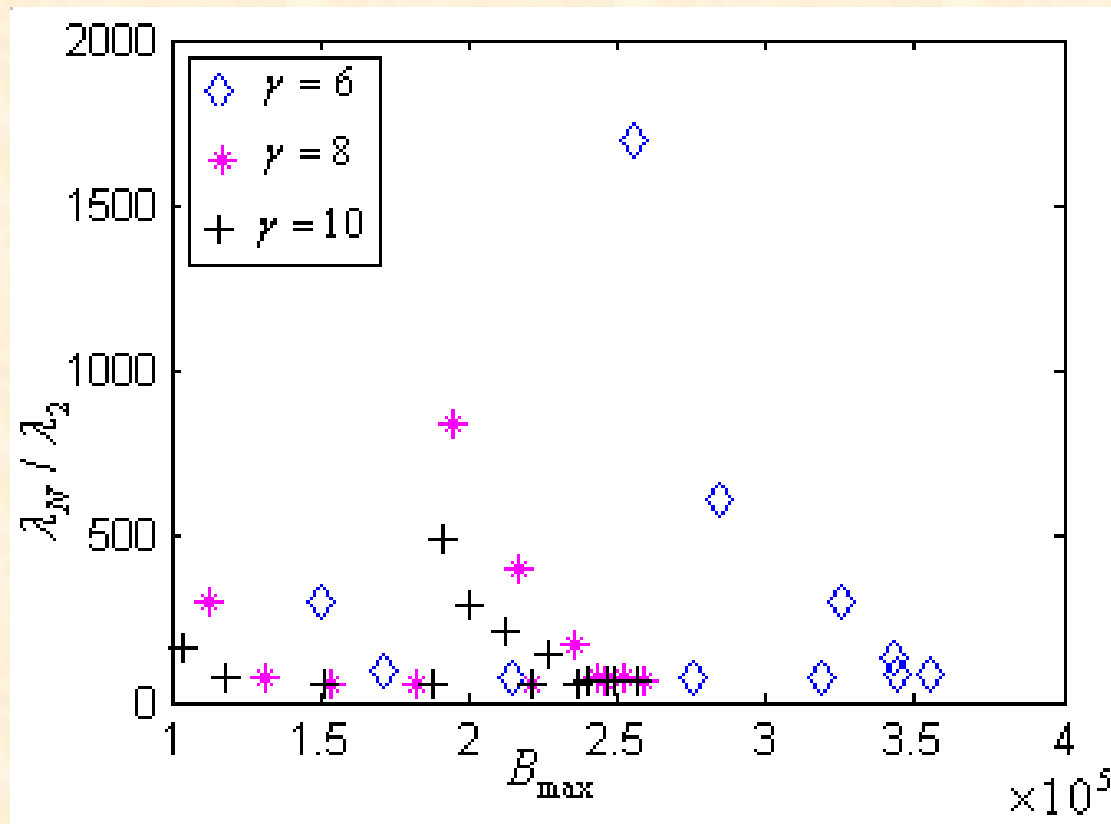
Hong et al. (2004)

Betweenness versus Synchronizability



Homogeneous Networks –
there is a clear
correlation
between
betweenness and
synchronizability

Betweenness versus Synchronizability



Heterogeneous Networks –
there is no clear correlation between betweenness and synchronizability for scale-free networks with large exponents in power-law

Enhancing the Network Synchronizability

Two examples:

- ❖ Perturbing the network structure

To eliminate the maximal betweenness

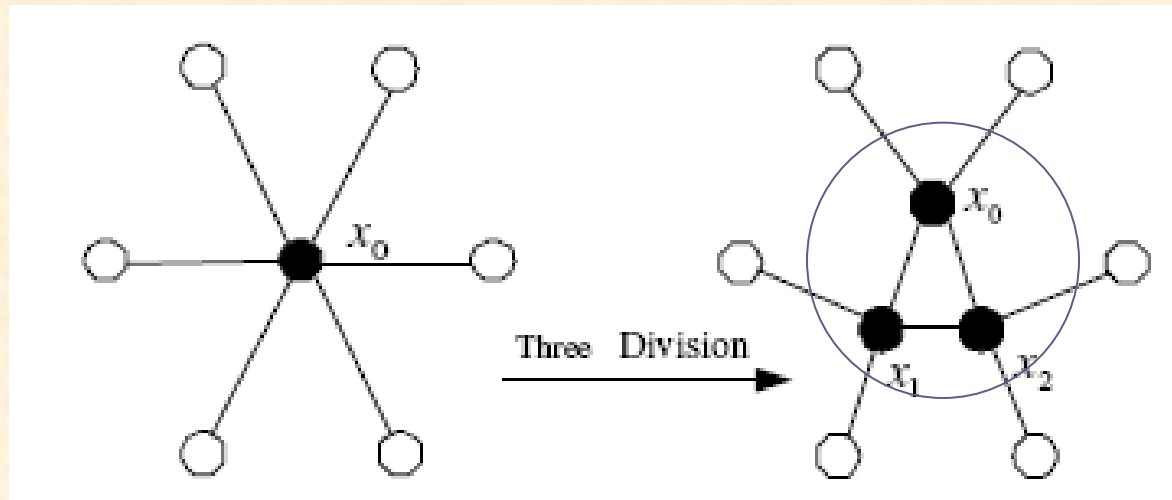
- ❖ Modifying the coupling structure

To reduce the impact of the heterogeneity of degree and betweenness distributions

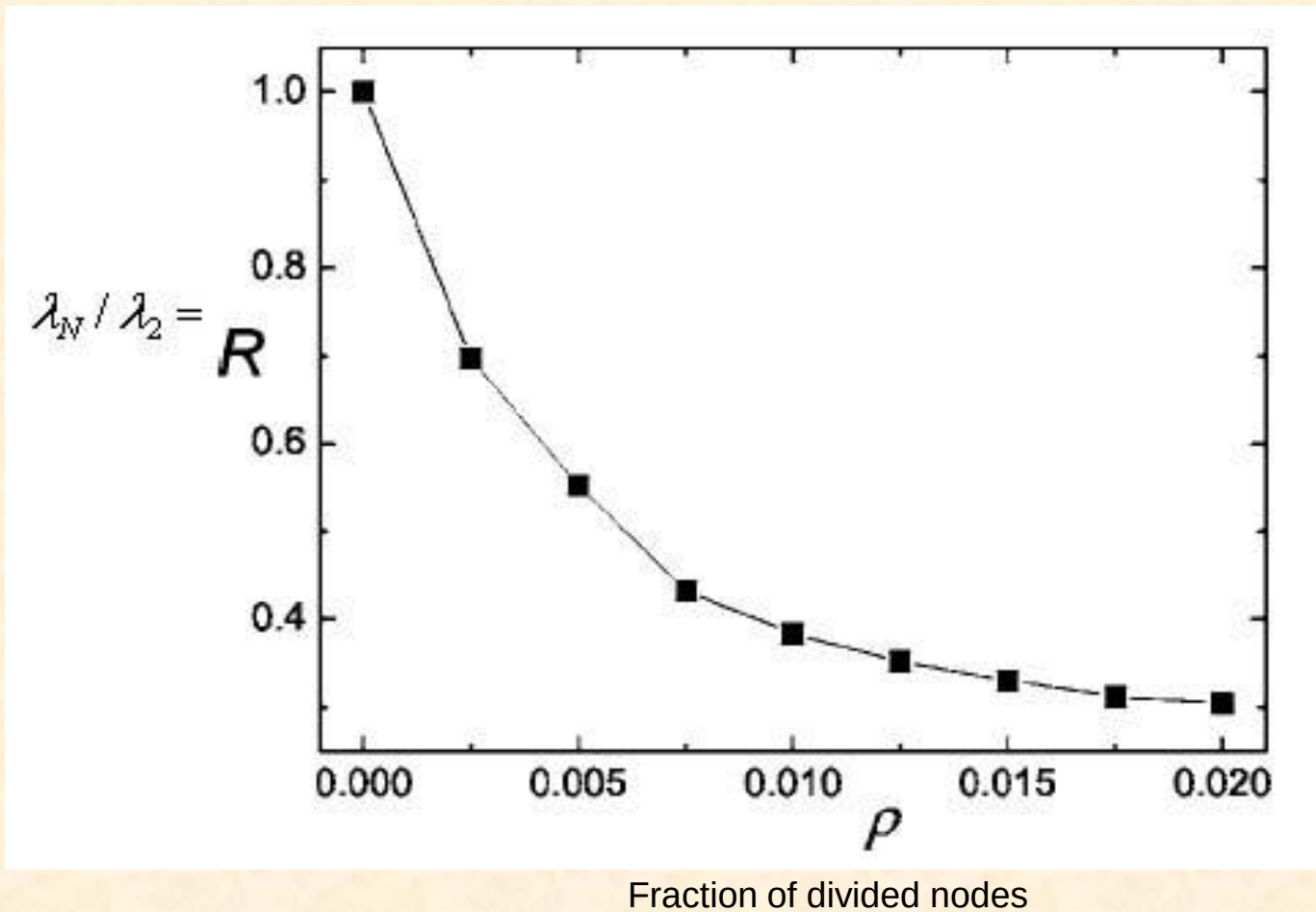
Perturbing the Network Topology

(Zhao, et al. Phys. Rev. E 2005, 72: 057102)

The node with the largest betweenness is replaced by several connected nodes, so that the shortest paths that passed through the original node will now only pass one or two new nodes, which will reduce the maximal betweenness dramatically.



Simulation Results



Decoupling Method (scale-free networks)

(CY Yin, WX Wang, G Chen, BH Wang, Phys. Rev. E 2006, 74: 047102)

Not only the nodes with large betweenness, but also the edges with large loads can cause data-traffic congestion.

Hence, if such heavily-loaded edges are decoupled, the data-traffic will be redistributed so become more efficient.

Algorithm:

1. Define and calculate the significance of all edges (i,j) by $k_i k_j$
2. Rank the edges from large to small
3. At each time step, cut an edge with the highest rank (which will decouple two nodes of higher degrees)

Simulation Results

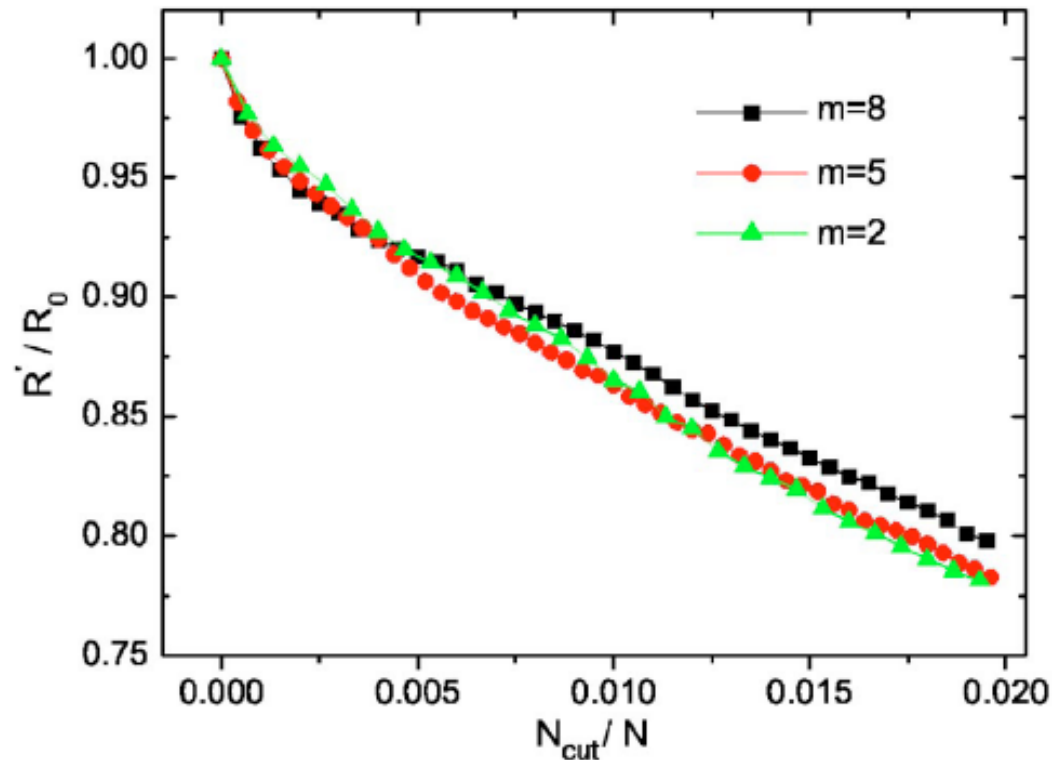


FIG. 1. (Color online) The change of synchronizability as a function of the proportion of cut edges N_{cut}/N for different values of m . Network size $n=2000$.

For scale-free networks, smaller eigen-ratio R'/R_0 implies better synchronizability, where m is the number of new edges in the scale-free networks generation

BREAK

10 minutes

Graph-Theoretic Approach

- ❖ Relationships between structural parameters and the synchronizability of complex networks
- ❖ Relationships between graph theory and the synchronizability of complex networks
- ❖ Disconnected synchronized regions
- ❖ Enhancing the network synchronizability

Structure versus Synchronizability

❖ Network Model:

$$\dot{x}_i = f(x_i) + c \sum_{j=1}^N a_{ij} H(x_j), \quad i = 1, 2, \dots, N \quad (1)$$

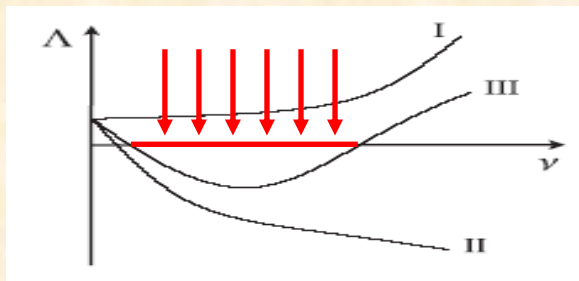
❖ Laplacian matrix L : symmetric, diffusive and irreducible, with eigenvalues

$$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N \quad (2)$$

❖ Synchronization of network (1) :

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t)\|_2 = 0, \quad i, j = 1, 2, \dots, N \quad (3)$$

- ❖ Synchronized region S : the region S of negative real α over which the network synchronizes.
- ❖ Remark 1: The synchronized region S can be an unbounded region, a bounded region, an empty set, or a union of several disconnected regions.
- ❖ Stability condition: The synchronous solution of network (1) is locally asymptotically stable: $c\lambda_k \in S, k = 2, 3, \dots, N$



synchronizing if $c\lambda_2 \in S_1$

Or if $\frac{\lambda_2}{\lambda_N} \in S_2$

❖ Remark 2: Key factors influencing the synchronizability:

- (i) inner-coupling matrix H
- (ii) eigenvalues of outer-coupling matrix A

❖ Synchronizability in terms of (ii):

1. bounded region (M. Banahona and L. M. Pecora, 2002)

$$\lambda_2 / \lambda_N, \quad 0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$$

2. unbounded region (X. F. Wang and G. Chen, 2002)

$$c\lambda_2, \quad 0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$$

3. union of several disconnected regions

(A. Stefanski, P. Perlikowski, and T. Kapitaniak, 2007)

(Z. S. Duan, C. Liu, G. Chen, and L. Huang, 2007)

Recall:

$$\dot{x}_i = f(x_i) + c \sum_{j=1}^N a_{ij} H(x_j), \quad i = 1, 2, \dots, N \quad (1)$$

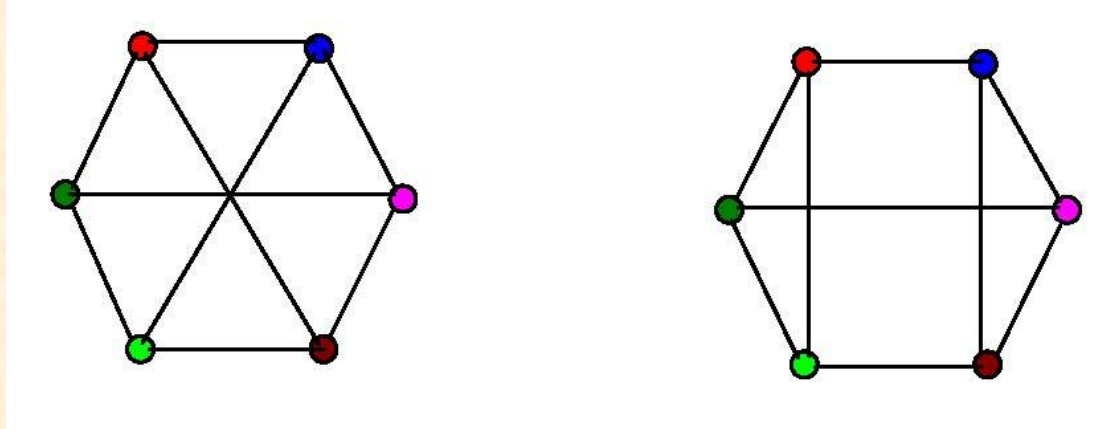
- ❖ Structural parameters that may affect the synchronizability:

average distance, degree distribution, clustering coefficient, node betweenness, ...

- ❖ Question:

How does synchronizability depend on various structural parameters?

Two simple graphs:



Graph G_1

Graph G_2

- ❖ They have the same structural characteristics:
Graphs G_1 and G_2 have
 - the same degree sequence: all node degrees: 3
 - the same average distance: $7/5$
 - the same node betweenness centrality: 2

- ❖ But they have different synchronizabilities:
eigenvalues of G_1 are 0, 3, 3, 3, 3 and 6
eigenvalues of G_2 are 0, 2, 3, 3, 5 and 5

$$\lambda_2(G_1) = 3 > \lambda_2(G_2) = 2$$

Let $\gamma = \lambda_2 / \lambda_N$ Then

$$\gamma(G_1) = 0.5 > \gamma(G_2) = 0.4$$

About edge-adding:

Lemma 1: For any given connected undirected graph G , all its nonzero eigenvalues increase monotonically with the number of added edges, i.e., by adding any edge e , one has $\lambda_i(G + e) \geq \lambda_i(G)$.

Lemma 1 → If the synchronized region is unbounded, then adding edges never decrease the synchronizability.

Main reason: $\lambda_i(G + e) \geq \lambda_i(G)$

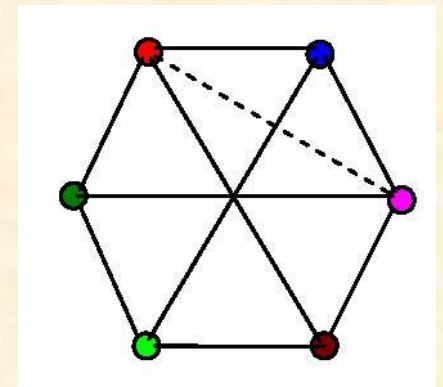
However, for bounded synchronized regions, this is not necessarily true because the eigen-ratio can either increase or decrease

$$\frac{\lambda_2}{\lambda_N} \rightarrow \downarrow$$

$$\frac{\lambda_2}{\lambda_N} \rightarrow \uparrow$$

Example: Adding an edge between red node and pink node in the graph G_2 : $e\{\text{red}, \text{pink}\}$ leads to a new graph $G_2 + e\{\text{red}, \text{pink}\}$, whose eigenvalues are 0, 2.2679, 3, 4, 5 and 5.7321, with $\gamma(G_2 + e\{\text{red}, \text{pink}\}) = 0.3956$, smaller than

$$\gamma(G_2) = 0.4$$



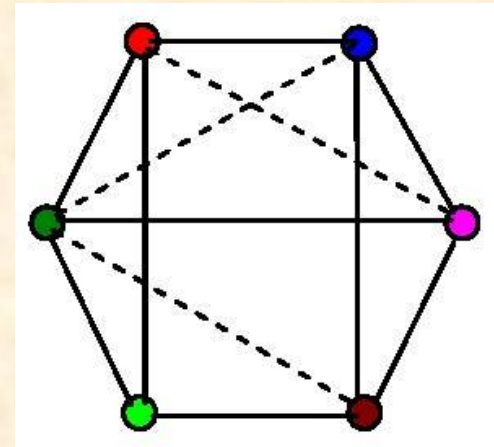
❖ **Example:** what about betweenness?

Consider the new graph

$$G = G_1 + e\{\text{red}, \text{pink}\} + e\{\text{blue}, \text{green}\} + e\{\text{green}, \text{brown}\}$$

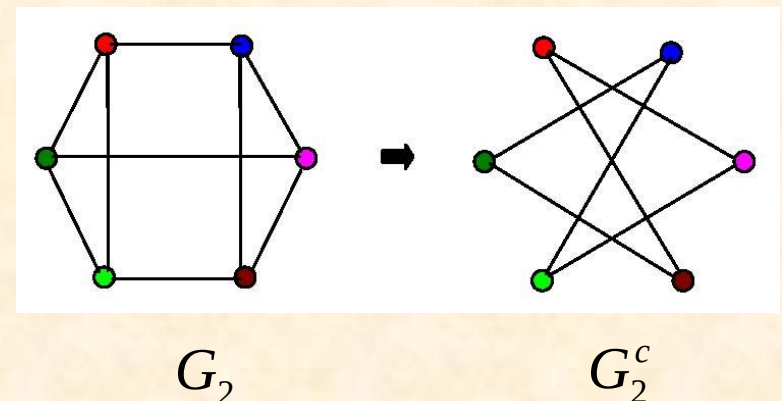
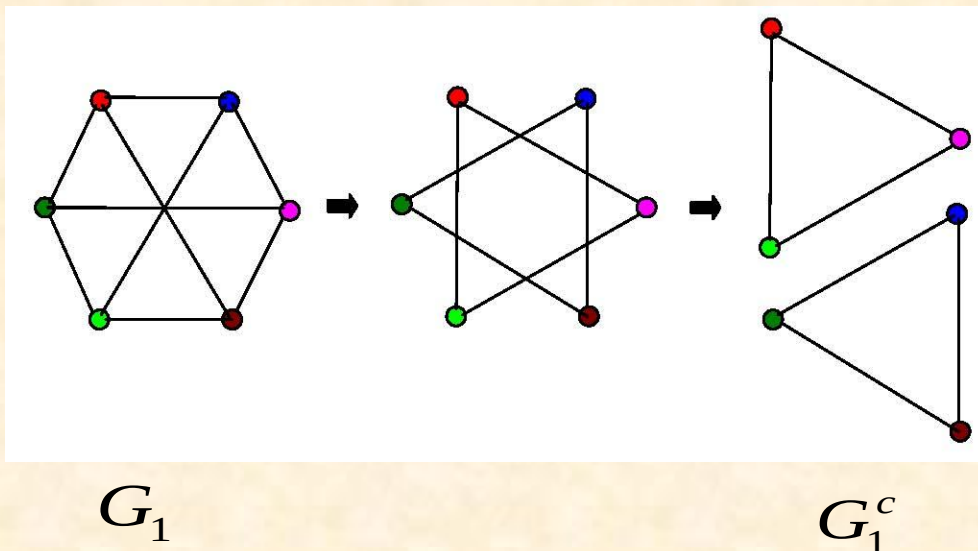
Maximum node **betweenness** centrality of G is $\frac{11}{6}$ smaller than that of graph G_1 , 2, but they have the same synchronizability.

→ The synchronizability may have other (yet unknown) relationships to the common network structural parameters.



Complementary Graphs: particularly when they are disconnected

- ❖ For a given graph G , the complement of G is the graph containing all the nodes of G , and the edges that are not in G , denoted by G^c
- ❖ The complements of G_1 and G_2 :



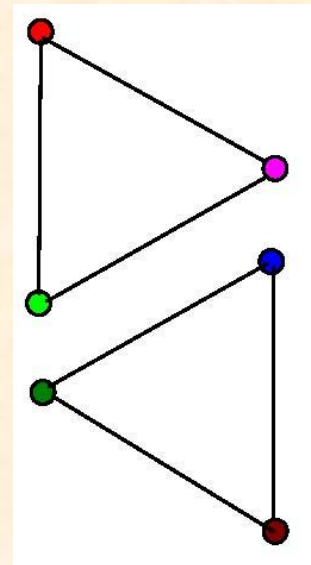
❖ **Lemma 2:** For any given graph G :

(i) the largest eigenvalue of G , $\lambda_N(G)$, satisfies $\lambda_N(G) \leq N$

(ii) $\lambda_N(G) = N$ if and only if G^c is disconnected. Moreover, if G^c has (exactly) q connected components, then the multiplicity of $\lambda_N(G) = N$ as an eigenvalue of G is $q-1$

(iii) $\lambda_i(G^c) = N - \lambda_{N-i+2}(G)$, $2 \leq i \leq N$

❖ G_1^c is disconnected. The largest eigenvalue of G_1 is 6, which remains the same for the graph with any more edges being added. Hence, Lemma 1 $\rightarrow$ the synchronizability of all the networks built on graph G_1 never decrease with edge-adding.



Recall: G_1 has eigenvalues: 0, 3, 3, 3, 3, 6

- ❖ **Remark 3:** With the same number of edges, generally the synchronizability of a network built on graph G is better when G^c has two disconnected components than the case that G^c has more than two disconnected components, since in the latter case, the multiplicity of the largest eigenvalue of G is larger than 1, namely, some edges there have no contributions to the synchronizability.

G^c

- ❖ **Remark 4:** Better understanding and careful manipulation of complementary graphs are useful for enhancing network synchronizability.

Disconnected Synchronized Regions of complex dynamical networks

❖ Typical types of synchronized regions:

- Type I: empty
- Type II: unbounded $S_1 = (\alpha_1, \infty)$
- Type III: bounded $S_2 = (\alpha_1, \alpha_2)$
- Type IV: union of some of the above
 $(\alpha_1, \alpha_2) \cup (\alpha_3, \infty)$ $(\alpha_1, \alpha_2) \cup (\alpha_3, \alpha_4) \cup (\alpha_5, \alpha_6)$ and so on

A. Stefanski, P. Perlikowski, and T. Kapitaniak, (2007)

Z. Duan, G. Chen, and L. Huang, (2007)

- ❖ When the synchronous state is an equilibrium, the synchronized region problem reduces to a stability problem of the matrix pencil $F + \alpha H$ with respect to parameter α .

(F is the Jacobian and H is the inner-linking matrix).

$$\dot{x}_i = f(x_i) + c \sum_{j=1}^N a_{ij} H(x_j), \quad i = 1, 2, \dots, N \quad (1)$$

- ❖ **Theorem:** For any natural number n , there are two matrices F and H of order $2(n-1)$ such that $F + \alpha H$ has n disconnected stable regions with respect to parameter α .

Examples of networks with disconnected synchronized regions

❖ **Example 3:** Consider network

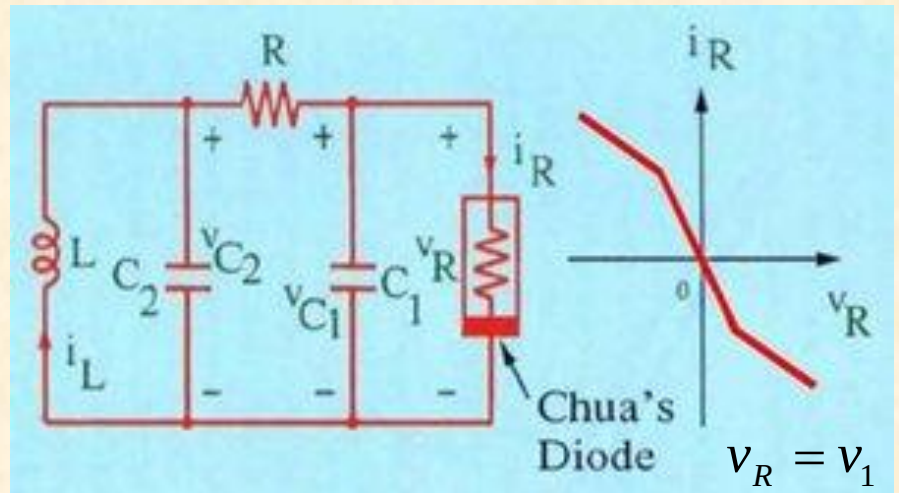
$$\dot{x}_i = f(x_i) + c \sum_{j=1}^N a_{ij} H(x_j), \quad i = 1, 2, \dots, N \quad (1)$$

with third-order smooth Chua's circuits, namely,

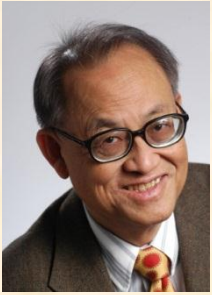
$$\begin{aligned} \dot{x}_{i1} &= -k\alpha x_{i1} + k\alpha x_{i2} - k\alpha(ax_{i1}^3 + bx_{i1}) \\ \dot{x}_{i2} &= kx_{i1} - kx_{i2} + kx_{i3} \\ \dot{x}_{i3} &= -k\beta x_{i2} - k\gamma x_{i3} \end{aligned} \quad (2)$$

Chua's Circuit:

$$\begin{cases} \frac{dv_1}{dt} = \frac{1}{C_1} [G(v_2 - v_1) - f(v_1)] \\ \frac{dv_2}{dt} = \frac{1}{C_2} [G(v_1 - v_2) + i_3] \\ \frac{di_3}{dt} = -\frac{1}{L} [v_2 + R_0 i_3] \end{cases}$$

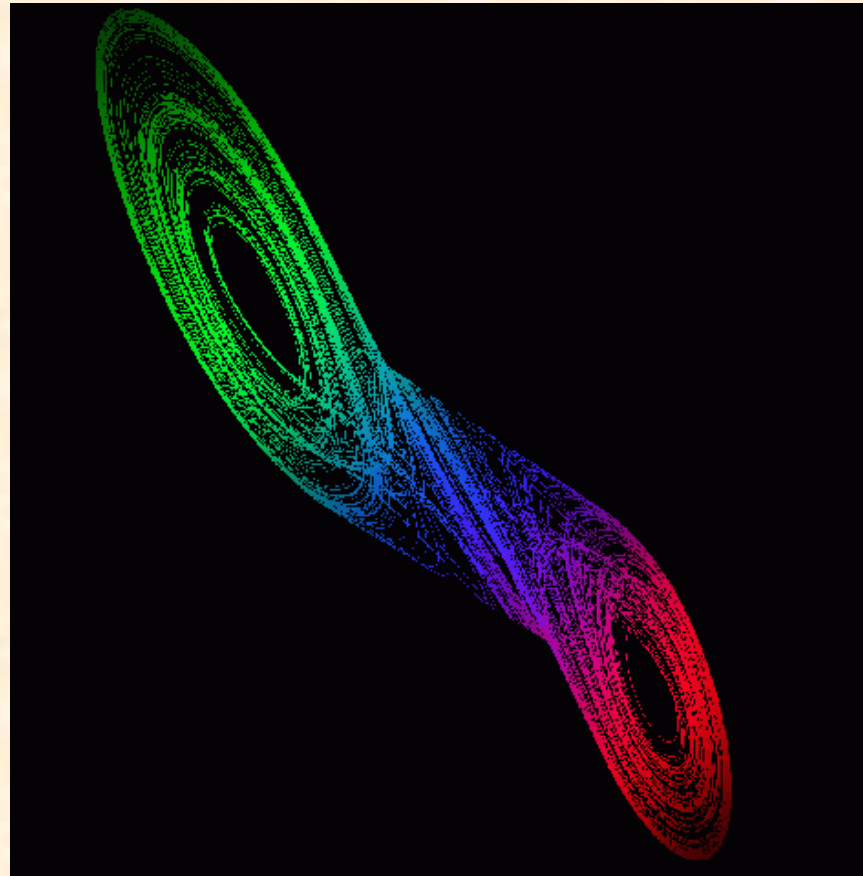
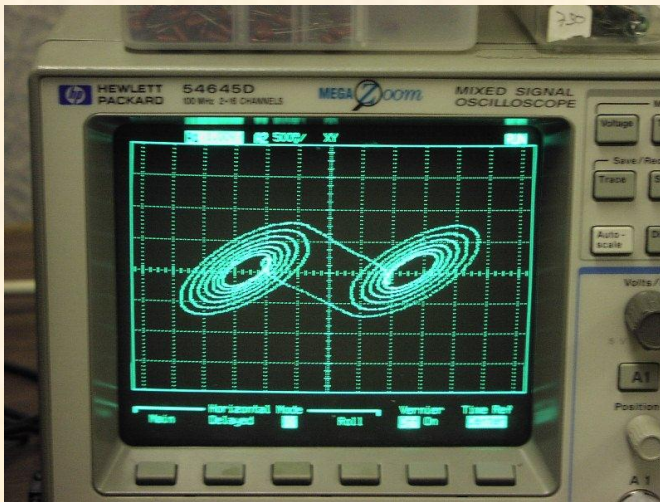


$$f(v_1) = G_b v_1 + \frac{1}{2} (G_a - G_b) \{|v_1 + E| - |v_1 - E|\}$$



Leon O Chua
(1936 -)

Chua's attractor



Linearizing (2) at the zero equilibrium yields

$$dx_i / dt = Fx_i, \quad F = \begin{pmatrix} -k\alpha - k\alpha b & k\alpha & 0 \\ k & -k & k \\ 0 & -k\beta & -k\gamma \end{pmatrix}$$

Take $k = 1$, $\alpha = -0.1$, $\beta = -1$, $\gamma = 1$, $a = 1$, $b = -25$, and

$$H = \begin{pmatrix} 0.8348 & 9.6619 & 2.6591 \\ 0.1002 & 0.0694 & 0.1005 \\ -0.3254 & -8.5837 & -0.9042 \end{pmatrix}$$

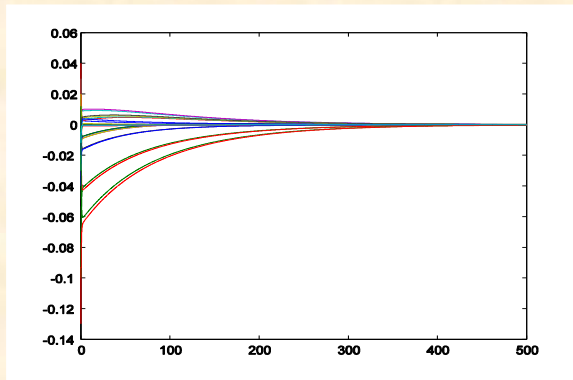
Then $F + \alpha H$ has a disconnected stable region:

$$S = [-0.0099, 0] \cup [-2.225, -1)$$

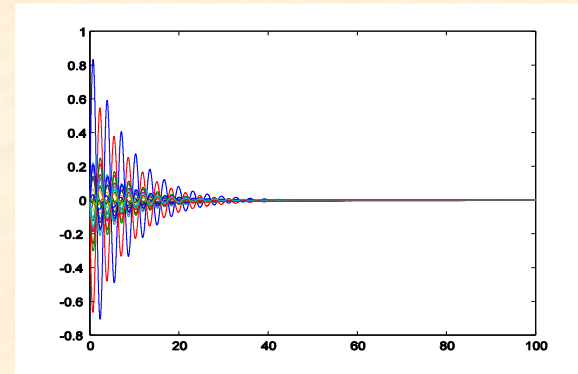
- ❖ Suppose that the number of nodes of network (1) is $N = 10$ and the outer coupled matrix A is a globally coupled matrix. Then network (1) with the above data achieves local synchronization when the coupling strength c satisfies

$$c \in [0, 0.00099] \quad \text{or} \quad c \in (0.1, 0.2225]$$

- ❖ Figures below visualize the synchronization and non-synchronization behaviors of this network.

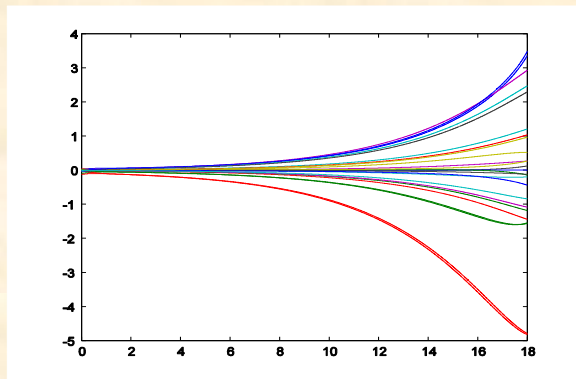


(a) $c = 0.0005 \in [0, 0.00099]$.

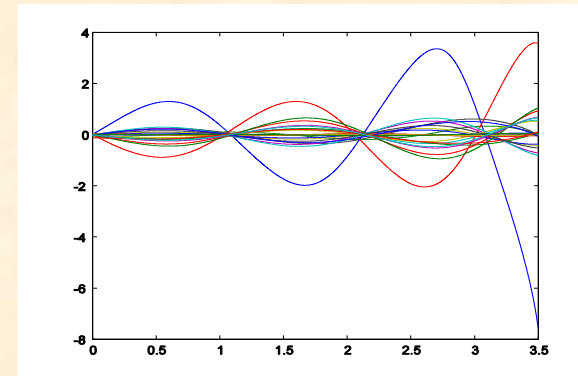


(b) $c = 0.2 \in (0.1, 0.2225]$.

Fig. 1 Synchronization of network (1) with different coupling strengths.



(c) $c = 0.02 \in (0.001, 0.1)$.



(d) $c = 0.3 \in (0.2225, +\infty)$.

Fig. 2 Non-synchronization of network (1) with different coupling strengths.

- ❖ **Example 4:** An interesting example of the coexistence of unbounded and bounded synchronization regions in the form of $S = (-\infty, -\alpha_1) \cup (-\alpha_2, -\alpha_3)$

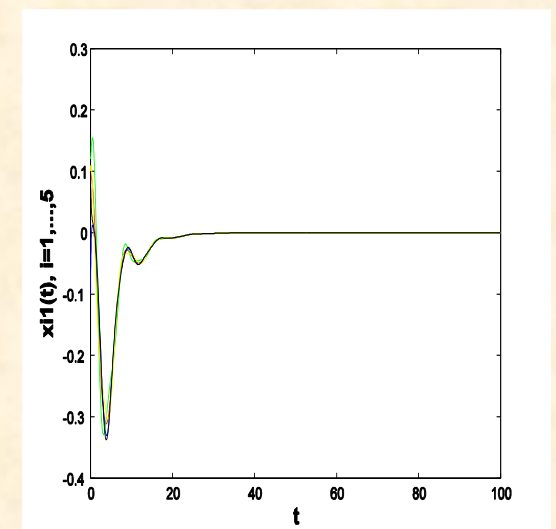
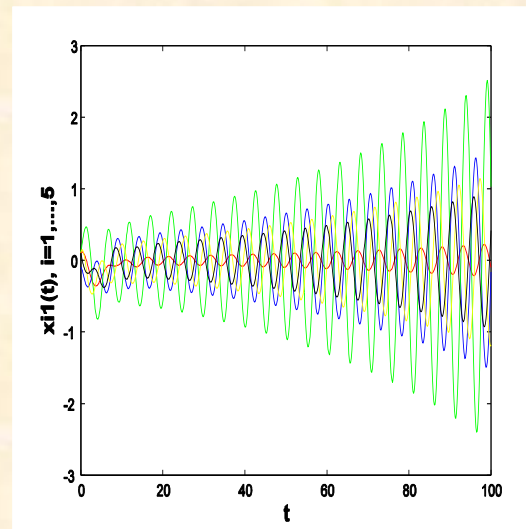
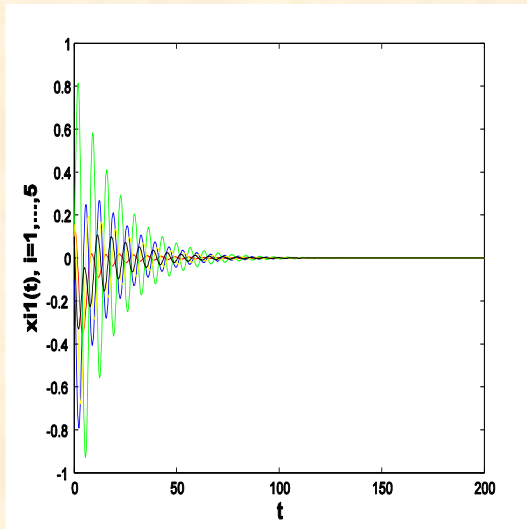
Consider a linearly and diffusively coupled dynamical network with the state equations

$$\dot{x}_i = Fx_i + c \sum_{j=1}^N a_{ij} Hx_j, \quad i = 1, 2, \dots, N, \quad (3)$$

where

$$F = \begin{bmatrix} \frac{1}{4} & 0 & 1 \\ 0 & \frac{7}{16} & 1 \\ 2 & -3 - \frac{7}{16} \cdot \frac{23}{16} & -\frac{23}{16} \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- ❖ The synchronization region of network (3) with is $S = \left(-\infty, \frac{1}{4}(-3 - \sqrt{2})\right) \cup \left(\frac{1}{4}(-3 + \sqrt{2}), 0\right]$
- ❖ Suppose that the number of nodes of network (3) is $N = 5$ and the outer coupled matrix A is a globally coupled matrix. Then, network (3) achieves local synchronization when the coupling strength c satisfies $c \in [0, 0.07929)$ or $c \in (0.22071, +\infty)$.
- ❖ Figures below visualize the synchronization and non-synchronization behaviors of this network.



(a) $c = 0.04 \in [0, 0.07929)$. (b) $c = 0.14 \in (0.07929, 0.22071)$. (c) $c = 1 \in (0.22071, +\infty)$.

Fig. 3 Synchronization and non-synchronization of the first state variables of the five nodes in network (3) in the global coupling configuration: $x_{i1}(t), i = 1, \dots, 5$.

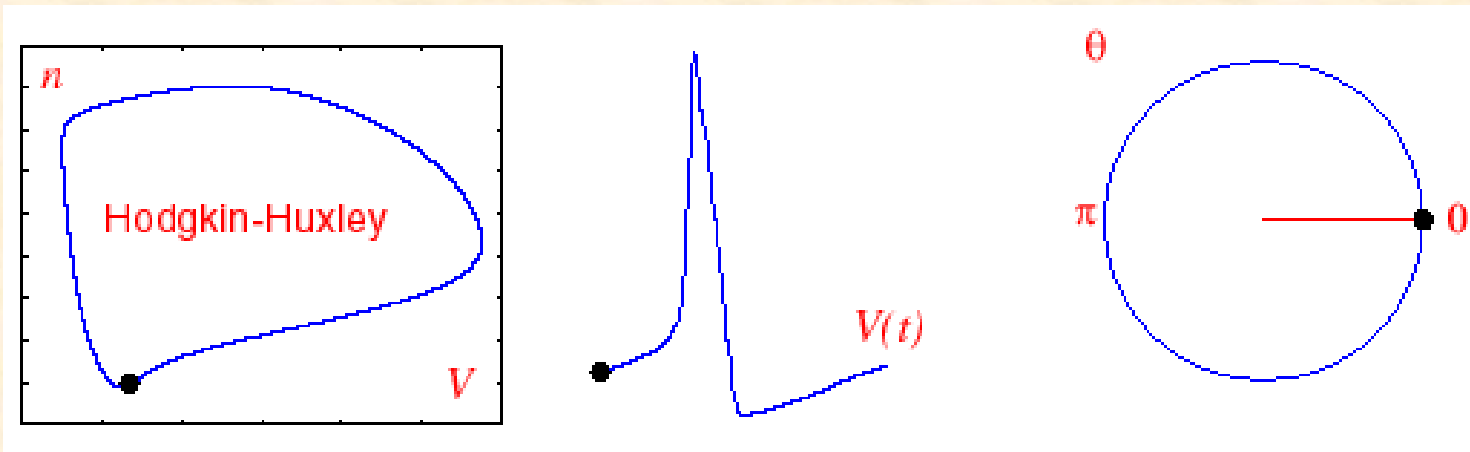
More ...

Phase Synchronization

Assume each node is a periodic oscillator

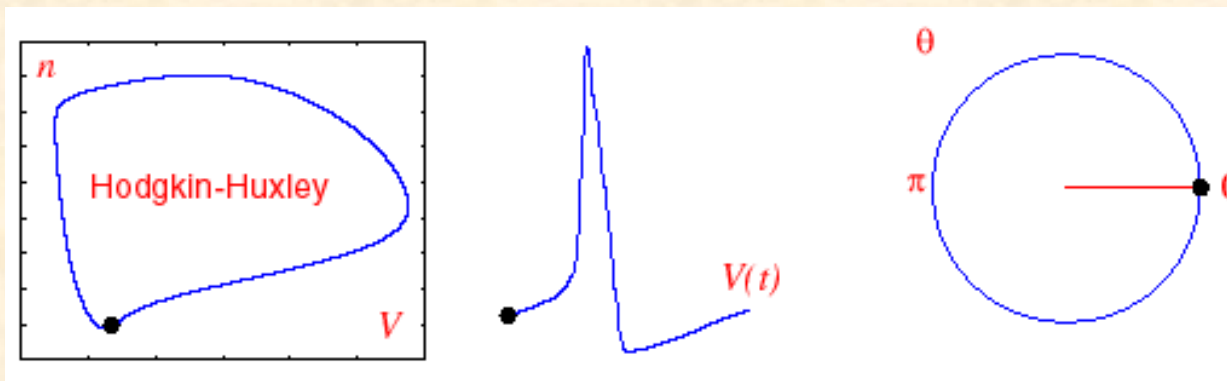
$$\dot{x} = f(x)$$

Examples: Van der Pol, Hodgkin-Huxley, Duffing oscillator...

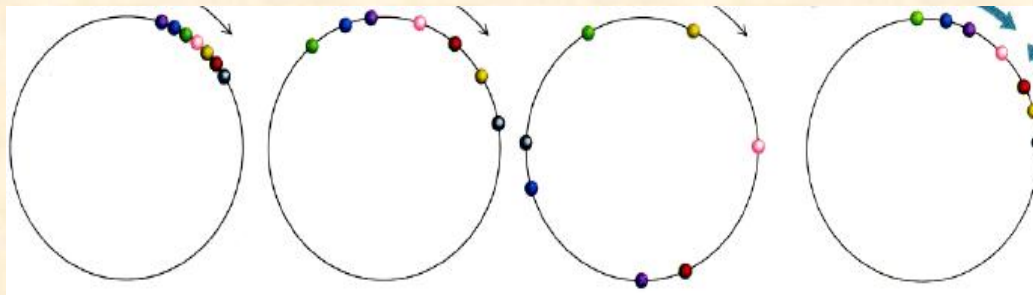


A canonical model for periodic oscillator

$\dot{\theta} = \omega$ where θ is the phase and ω is the frequency



Simplicity
Universality
Limitation



Winfree's Pioneering Work

$$\dot{\theta}_i = \omega_i + \left(\sum_{j=1}^N X(\theta_j) \right) Z(\theta_i)$$

- ❖ When the spread of natural frequencies is large compared to the coupling, the system behaves incoherently, with each oscillator running at its natural frequency.
- ❖ As the spread is decreased, the incoherence persists until a certain threshold is crossed —then a small cluster of oscillators suddenly freezes into synchrony.

A. T. Winfree, *The Geometry of Biological Time*, Springer, NY, 1980.

Kuramoto Model

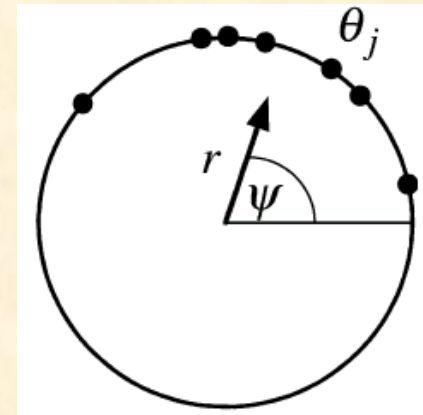
$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i)$$

- ◆ Here, K is the coupling strength and the factor $1/N$ ensures that the model is well behaved as $N \rightarrow \infty$
- ◆ The frequencies ω_i are distributed according to some probability density $g(\omega)$
- ◆ For simplicity, Kuramoto assumed that $g(\omega)$ is unimodal and symmetric about its mean frequency

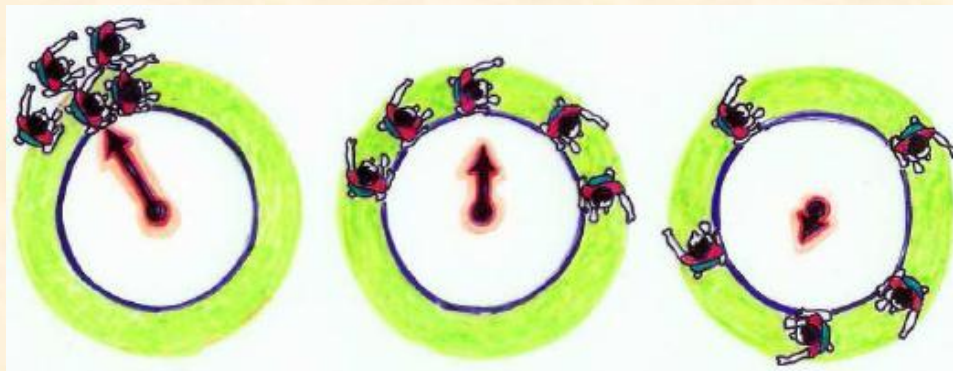
Order Parameter

- ❖ To visualize the dynamics of the phases, it is convenient to imagine a group of points running around the unit circle in the complex plane:

$$re^{i\psi} = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}$$



$r \approx 1$

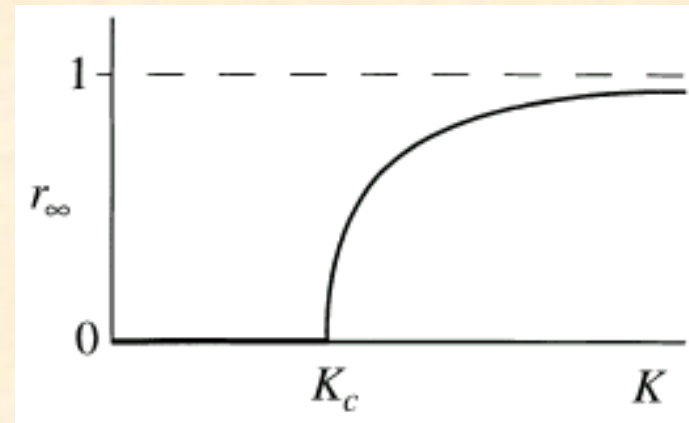
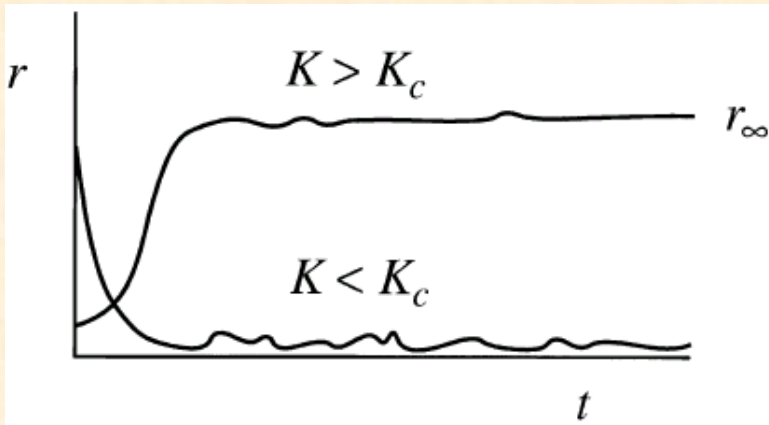


$r \approx 0$

Synchronization Threshold

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i)$$

$$r e^{i\psi} = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}$$



J. A. Acebrón, et al: Rev. Mod. Phys. 77 (2005) 137

End

